$\mathcal{M}$-Completeness, Smith Spaces, & Liquidity

Recall:

S is finite

$$\mathcal{M}(S)_{\le 1} := \{ \sum a_i s_i \mid \sum |a_i| \le 1 \} \subseteq \mathbb{R}[S]$$

$$\mathcal{M}(S)_{\le c} := \{ \sum a_i s_i \mid \sum |a_i| \le c \} \subseteq \mathbb{R}[S]$$

$S = \lim S_i$ profinite

$$\mathcal{M}(S)_{\le 1} := \varprojlim \mathcal{M}(S_i)_{\le 1}$$

$$\mathcal{M}(S)_{\le c} := \varprojlim \mathcal{M}(S_i)_{\le c}$$

$$\mathcal{M}(S) = \bigcup_{c \ge 0} \mathcal{M}(S)_{\le c} \longleftarrow \text{signed Radon measures.}$$

Fact

$$\mathcal{M}(S) \longleftrightarrow \left\{ \begin{array}{l} \mu : \left\{ \begin{array}{l} \text{open \& closed} \\ \text{subs of } S \end{array} \right\} \longrightarrow \mathbb{R} \quad \text{s.t.} \\ ① \ \mu(U \amalg V) = \mu(U) + \mu(V) \\ ② \ \exists\, c_\mu \ \text{ s.t. } \ \forall\ S = S_1 \amalg \cdots \amalg S_n \\ \qquad \sum |\mu(S_i)| \le c_\mu \end{array} \right\}$$

Prop V a Banach space. $S \in \mathrm{PFSet}$.

$$\begin{tikzcd}
S \arrow[r, "f"] \arrow[d, "\delta"'] & V \\
\mathcal{M}(S) \arrow[ur, dashed, "\exists \tilde{f}"] & \\
\mu \arrow[r, mapsto] & \int f \, d\mu
\end{tikzcd}$$

Cor: V Banach $f : S \longrightarrow V$ cts then f factors over a compact absolutely convex subset of V. ~~[illegible] the im f.~~

$(\tilde{f}(\mathcal{M}(S)_{\le 1}))$.

Condensed version of a Banach space (ie. complete space).

Def

A condensed $\mathbb{R}$-vector space that is quasi-separated is $\mathcal{M}$-complete if $\forall$ cts

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d] & V \\ \mathcal{M}(S) \arrow[ur, dotted] & \end{tikzcd}$$

Cor.

Banach spaces are $\mathcal{M}$-complete.

(U)
Prop: The extension $\tilde{f}$ is necessarily unique.

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§ What Can We Detect from topology?

Lemma

X qsep

$$\{\text{qcpt inj } Z \hookrightarrow X\} \longleftrightarrow \left\{\begin{matrix}\text{closed subs} \\ W \subseteq X(*)\end{matrix}\right\}$$

$$Z \longmapsto Z(*)$$

$$X \times_{X(*)} W \longleftarrow\!\!\shortmid W$$

Pf/ Reduce to X qcpt by colims. Then

$$\{\text{qc qs cond}\} \longleftrightarrow \left\{\begin{matrix}\text{comp} \\ \text{Hausdorff}\end{matrix}\right\}$$

$$\text{inj} \longleftrightarrow \text{closed imm.}$$

Proof of prop $\mathcal{U}$

$$\begin{tikzcd} S \subseteq \mathcal{M}(S) \arrow[r, "g"] \arrow[r, bend right, "0"'] & V \end{tikzcd} \quad \leftarrow \text{qsep \& comp.}$$

$$\begin{tikzcd} g^{-1}(0) \arrow[r, hook] & \mathcal{M}(S) \end{tikzcd}$$
$$\cup\!\text{I}$$
$$S$$

qcpt inj so in top sense closed sub.

But S dense in $\mathcal{M}(S)_{\leq 1}$

So $g^{-1}(0) \supseteq \mathcal{M}(S)_{\leq 1}$

So by linearity it contains $c\mathcal{M}(S)_{\leq 1} = \mathcal{M}(S)_{\leq c}$ $\forall c$.

Cor: One can check $\mathcal{M}$-completeness on ED Sets.

Pf/

$$\begin{tikzcd} S_1 \arrow[r, shift left] \arrow[r, shift right] & S_0 \arrow[r] & S \arrow[r] & V \end{tikzcd}$$

$S_1 = \beta(S_0 \times_S S_0)^{disc}$, $S_0 = \beta S^{disc}$

& $Hom(S, V) = \ker\left(Hom(S_0, V) \rightrightarrows Hom(S_1, V)\right)$

$$\begin{tikzcd} S_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & S_0 \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & V \\ \mathcal{M}(S_1) \arrow[r, shift left, "!"] \arrow[r, shift right, "!"'] \arrow[urrr, bend right, "\exists!"'] & \mathcal{M}(S_0) \arrow[r, "!"] \arrow[urr, bend right, "\exists!"'] & \mathcal{M}(S) \arrow[ur, dotted, "\exists!?"] & \end{tikzcd}$$

Prop $\mathcal{M}(S)$ is $\mathcal{M}$-complete.

Pf

$$\begin{tikzcd}
T \arrow[rr] \arrow[rrd, dashed, "\exists!"] \arrow[rrdd, dashed] \arrow[rrddd] \arrow[dd, phantom, "\cap"] & & \mathcal{M}(S) \\
 & & \mathcal{M}(S)_{\leq c} \arrow[u, phantom, "\cup\vert"] \arrow[d, phantom, "="] \\
\mathcal{M}(S)_{\leq 1} \arrow[rru, dashed] \arrow[rrd] & & \lim \mathcal{M}(S_i)_{\leq c} \arrow[d] \\
 & & \mathcal{M}(S_i)_{\leq c}
\end{tikzcd}$$

$\exists!$ by convexity!

Recall:
Studied Solid by probing w/ $\mathbb{Z}[S]^{\blacksquare}$ which had nice properties allowed to understand internal categorical structure. Do the same here w/ $\mathcal{M}(S)$. Leads to the notion of....

Smith Spaces

Properties of $\mathcal{M}(S)$:

- Compact absolutely ~~closed~~ convex subset

$$K \quad \text{s.t.} \quad \mathcal{M}(S) = \bigcup_{c \geq 0} cK$$

$$K = \mathcal{M}(S)_{\leq 1} \quad \& \quad cK = \mathcal{M}(S)_{\leq c}.$$

Def^n (top version)

- A Smith Space is a complete locally convex top. $\mathbb{R}$ vect space admitting a convex compact $K \subset V$ s.t.
  $V = \bigcup_c cK$

Defn (cond version)

- A condensed $\mathbb{R}$-vector space V is a smith space if it is qsep, $\mathcal{M}$-complete, & $\exists$ compact Hausdorff $K \subseteq V$ w/ $V = \bigcup_c cK$

(note, makes sense by top lemma).

Seen: $\mathcal{M}(S)$ a smith space.

Prop: V a top $\mathbb{R}$ vect space $\Rightarrow$

V smith $\Leftrightarrow$ $\underline{V}$ is.

Pf/ Top lemma $\rightarrow$

$\Leftarrow$ $\underline{V}(*) = V = \bigcup_c cK(*)$

2 questions 1) K abs. convex? 2) $\mathcal{M}$-complete imply complete? | **Appendix**

Said earlier that we can probe Banach spaces or more generally any $\mathcal{M}$-complete space w/ Smith spaces. Let's prove this.

Prop V Banach ($\mathcal{M}$-complete).
Consider category of smith spaces $W \subseteq V$.
1) This category is filtered
2) $\underline{V} = \varinjlim \underline{W}$

Pf/ 1) $W_1, W_2 \subseteq V$.
K_1, K_2 cpt gen. subs.

$$\begin{tikzcd} K_1 \amalg K_2 \arrow[r] \arrow[dr, dashed] & V \\ & K \arrow[u, phantom, sloped, "\subseteq"] & \arrow[l] \text{cpt \& abs. convex.} \end{tikzcd}$$

K " $\left(\begin{array}{l}\text{limits of conv combos} \\ \text{of } K_1 \ \& \ K_2\end{array}\right)$

$W = U_c K$ smith & contains W_1 & W_2.

2) Show a map $S \longrightarrow V$ factors over some W.
But take $W = U_c K$ w/ $K = \overline{J}(\mathcal{M}(S)_{\leq c})$.

The Category of $\mathcal{M}$-Complete $\mathbb{R}$-Vect Spaces

Theorem $\mathcal{M}$-complete $\mathbb{R}$-vector spaces are pecisely filtered colims of Smith spaes along injections.

Pf/ $\Longrightarrow$ V $\mathcal{M}$-complete. Take

$$\begin{tikzcd} & S \arrow[rr] & & V & \\ \mathcal{M}(S) \supseteq & \mathcal{M}(S)_{\leq 1} \arrow[u, phantom, sloped, "\subseteq"] \arrow[rr] & & K \arrow[u, phantom, sloped, "\subseteq"] \subseteq W = U_c K & \\ \mathcal{M}(T) \supseteq \arrow[u] \arrow[ur] & \mathcal{M}(T)_{\leq 1} \arrow[u] \arrow[urr, dashed, "\exists"] & \supseteq & T \in \text{FDSet} \arrow[u] \arrow[ur, bend right] & \end{tikzcd}$$

Show $\mathcal{M}$-comp

$\wp$

$\Leftarrow$ Show $\operatorname{colim} W_i$ is qsep ~~quasicompact~~.

qsep preserved under filtered colims along injections

$$\begin{tikzcd} S \arrow[r] \arrow[d] \arrow[dr] & \operatorname{colim} W_i \\ & \cup\!\!\!\mid \\ \mathcal{M}(S) \arrow[r] & W_i \end{tikzcd}$$

Can understand $\mathcal{M}$-complete spaces in terms of Smith ones!

Nice Categorical Properties.

Prop (The Quasi-Separator)

The inclusion

$$\{\text{qsep Cond}\} \xrightarrow{\quad} \{\text{Cond}\}$$

with a dashed arrow $\{\text{Cond}\} \dashrightarrow \{\text{qsep Cond}\}$ labelled qs,

admits an adjoint.

Pf

$$X' = \coprod S_i \longrightarrow X \qquad S_i \text{ pf.}$$

$$R = X' \times_X X' \subseteq X' \times X' \quad \text{be induce eq. relation giving } X.$$

Suppose have

$$\begin{tikzcd} X \arrow[r] & Y \\ X' \arrow[u] \arrow[ur] \end{tikzcd} \quad w/ \ Y \text{ qsep.}$$

Get $X' \times_Y X \subseteq X' \times X'$ qcpt injection. Top lemma a closed subspace. Also contains R. Top lemma, makes sense to let $\overline{R}$ = minimal closed eq relation containing R.

$X^{qs} = X' / \overline{R}$. Works by construction.

~~Hom(X,Y)~~

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \\ X^{\beta S} \arrow[ur] & \end{tikzcd} \quad \rightsquigarrow \quad \begin{array}{c} R \subseteq X' \times_Y X' \\ \cap\!| \quad \cong \\ \bar{R} \end{array}$$

Theorem:

1) The category $\mathcal{M}$-complete admits kernels & cokernels } Same!

2) The image & kernel of a map between $\mathcal{M}$-complete condensed sets is $\mathcal{M}$-complete.

Pf/ Kernel $f: V \to V'$

$$\begin{tikzcd} S \arrow[r] \arrow[d] \arrow[rrr, bend left, "0"] & \ker f \arrow[r] & V \arrow[r, "f"] & V' \\ \mathcal{M}(S) \arrow[ur, dotted, "\exists!"] \arrow[urr, "\exists!"'] \arrow[urrr, "0"'] & & & \end{tikzcd}$$

Image: Noting V is colimit of $\mathcal{M}(S)$, may assume $V = \mathcal{M}(S)$.

$$\begin{array}{ccc} \mathcal{M}(S) & \xrightarrow{f} & V' \\ \cap\!| & & \cup\!| \\ \mathcal{M}(S)_{\leq 1} & \longrightarrow & K \subseteq W := \bigcup_c cK = \operatorname{im}(f) \end{array}$$

↑ $\mathcal{M}$-complete (smith!)

Cokernel

Let $Q = \mathrm{coker}_{\mathrm{cond}}(f)$

Show Q^{qs} works.

1) Show it is $\mathcal{M}$-complete.

$$\begin{tikzcd}
\mathrm{im}(f) = W \arrow[r, two heads] & Q \arrow[r, two heads] & Q^{qs} \\
\mathcal{M}(S) \supseteq S \in \mathrm{EDSet} \arrow[u, dashed, "\exists!"] \arrow[u, dashed, "\exists"'] \arrow[ru, dashed, "\exists"] \arrow[rru] & &
\end{tikzcd}$$

2) Show its cokernel.

$$\begin{tikzcd}
V \arrow[r] \arrow[rd, "0"'] & V' \arrow[r] \arrow[d] & Q \arrow[r] \arrow[ld, "\exists!"] & Q^{qs} \arrow[lld, "\exists! \text{ by adj.}"] \\
& X & &
\end{tikzcd}$$

X $\mathcal{M}$-complete

Remarks

1) Sec not closed under cokernels, but it does have cokernels.

2) May hope that $\mathcal{M}$-complete w/ q sep may work, but this turns out to be false.

3) $\mathcal{M}\text{-Compl} \hookrightarrow \mathrm{Cond}/\mathbb{R}$, with left adjoint $\hat{\ }$

pf/ $\oplus \mathbb{R}[S_j] \to \oplus \mathbb{R}[S_i] \to V \to 0$

$\hat{V} = \mathrm{coker}\left(\oplus \mathcal{M}(S_j) \to \oplus \mathcal{M}(S_i)\right)^{qs}$