

Recall:

S is finite

$$M(S)_{\leq 1} := \{ \sum a_i s_i \mid \sum |a_i| \leq 1 \} \subseteq \mathbb{R}[S]$$

$$M(S)_{\leq c} := \{ \sum a_i s_i \mid \sum |a_i| \leq c \} \subseteq \mathbb{R}[S]$$

$S = \varprojlim S_i$ profinite

$$M(S)_{\leq 1} := \varprojlim M(S_i)_{\leq 1}$$

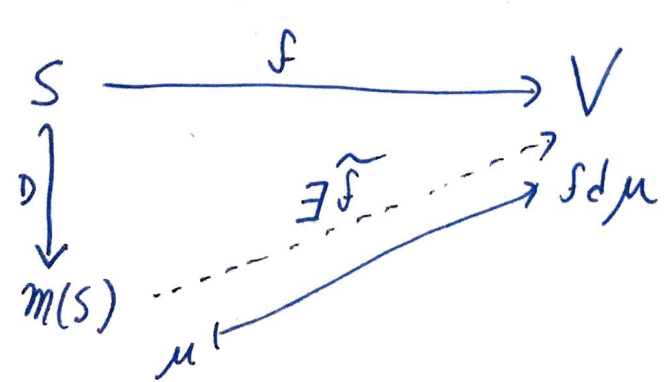
$$M(S)_{\leq c} := \varprojlim M(S_i)_{\leq c}$$

$$M(S) = \bigcup_{c > 0} M(S)_{\leq c} \longleftarrow \text{signed Radon measures.}$$

Fact

$$M(S) \longleftarrow \left\{ \begin{array}{l} \mu: \{ \text{open \& closed subs of } S \} \rightarrow \mathbb{R} \text{ s.t.} \\ \textcircled{1} \mu(u \sqcup v) = \mu(u) + \mu(v) \\ \textcircled{2} \exists c_\mu \text{ s.t. } \forall S = S_1 \sqcup \dots \sqcup S_n \\ \sum |\mu(S_i)| \leq c_\mu \end{array} \right\}$$

Prop V a Banach space. $S \in \text{PFSet}$.

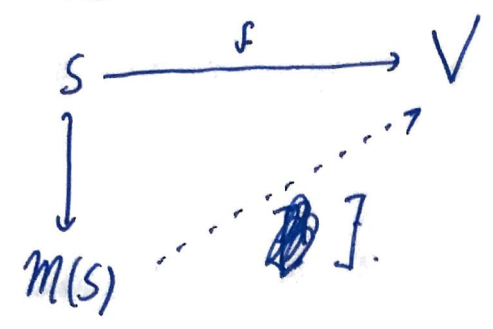


Cor: V Banach $f: S \rightarrow V$ cts fun f factors over a compact absolutely convex subset of V . ~~the int.~~
 $(\tilde{f}(M(S)_{\leq 1}))$.

Condensed version of a Banach space (i.e. complete space) - (2)

Defⁿ

A condensed \mathbb{R} -vector space that is quasi-separated is \mathcal{M} -complete
 iff V cts



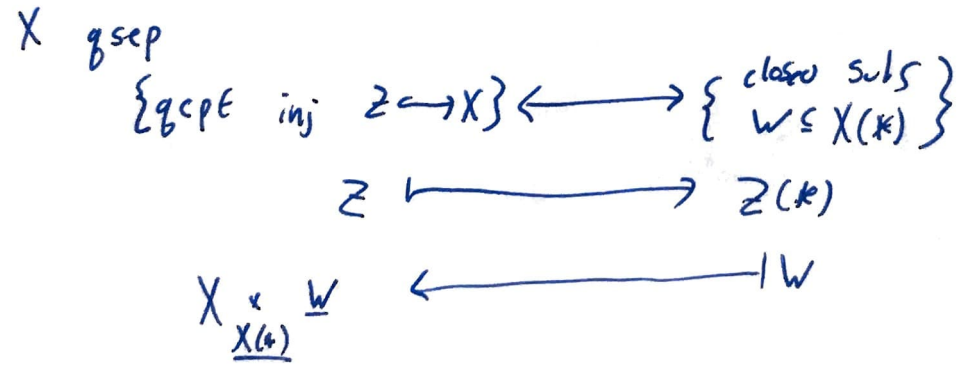
Cor.

Banach spaces are \mathcal{M} -complete.

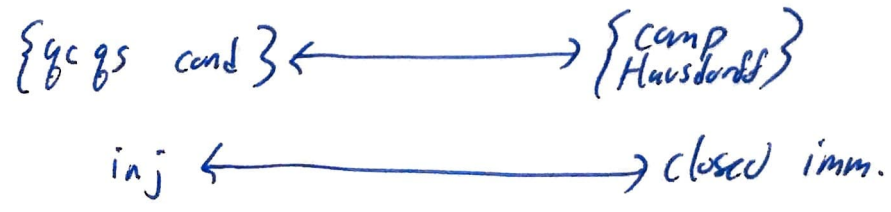
Prop: The extension \tilde{f} is necessarily unique.

§ What can we detect from topology?

Lemma



Pf/ Reduce to X qcpt by colms. Then.



Proof of prop 2

$$S \subseteq M(S) \xrightarrow{g} V$$

↘
0

← gcp & conl.

$$g^{-1}(0) \xrightarrow{\cup} M(S) \quad \text{gcp inj so in top sense closed sub.}$$

∪
S

But S dense in $M(S)_{\leq 1}$

$$\text{So } g^{-1}(0) \supseteq M(S)_{\leq 1}$$

$$\text{So by linearity it contains } cM(S)_{\leq 1} = M(S)_{\leq c} \quad \forall c.$$

Cor: One can check M-completeness on ED Sets.

PS/

$$S_1 \xRightarrow{\beta} S_0 \longrightarrow S \longrightarrow V$$

" "

$\beta(S_0 \xrightarrow{\beta} S_0)^{\text{disc}}$ βS^{disc}

$$\& \text{Hom}(S, V) = \ker(\text{Hom}(S_1, V) \xrightarrow{\beta} \text{Hom}(S_0, V) \xrightarrow{\beta} \text{Hom}(S, V))$$

$$\begin{array}{ccccccc} S_1 & \xRightarrow{\beta} & S_0 & \longrightarrow & S & \longrightarrow & V \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ M(S_1) & \xRightarrow{\beta} & M(S_0) & \longrightarrow & M(S) & \longrightarrow & V \end{array}$$

↘ ↘ ↘ ↘

Defn (top version)

5

• A Smith Space is a complete locally convex top. \mathbb{R} vect space admitting a convex compact $K \subset V$ s.t.

$$V = \bigcup_i K$$

Defn (cond version)

• A condensed \mathbb{R} -vector space V is a Smith space if it is \mathcal{G} sep, M -complete, & \mathbb{Z} compact Hausdorff

$$\mathbb{R} \subset V \quad \text{w/} \quad V = \bigcup_i K$$

(note, makes sense by top lemma).

Seen: $M(S)$ a Smith space.

Prop: V a top \mathbb{R} vect space \implies

$$V \text{ Smith} \iff \underline{V} \text{ is.}$$

PS Top lemma \implies

$$\longleftarrow V(K) = V = \bigcup_i K(*)$$

- 2 questions
- 1) K abs. convex?
 - 2) M -complete imply complete?

Appendix

Since earlier that we can probe Banach spaces or more generally some M -complete space w/ Smith spaces. Let's prove this.

Prop \forall Banach (M -complete).

(6)

Consider category of Smith spaces $W \subseteq V$.

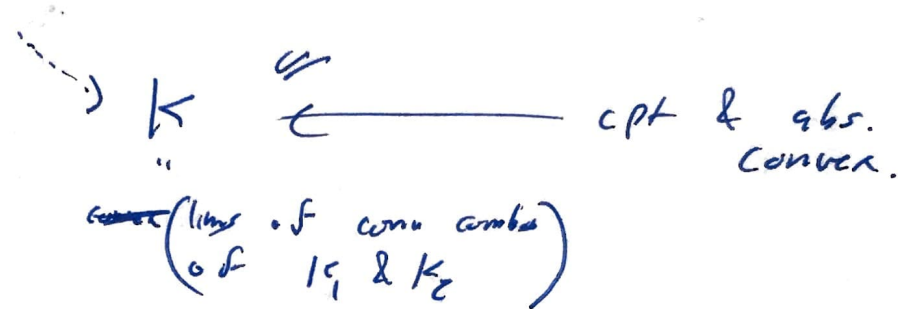
1) This category is filtered

2) $V = \varinjlim W$

Pf 1) $W_1, W_2 \subseteq V$.

K_1, K_2 cpt gen. subs.

$$K_1 \cup K_2 \longrightarrow V$$



$W = U \cup K$ Smith & contains W_1 & W_2 .

2) Show a map $S \rightarrow V$ factors over some W .

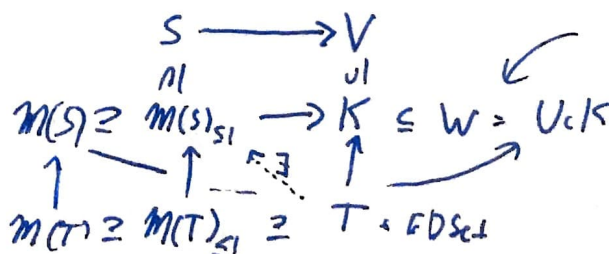
But take $W = U \cup K$ w/ $K = \overline{S}(\{M(S) \leq c\})$.

The Category of M -Complete \mathbb{R} -Vect Spaces

Theorem M -complete \mathbb{R} -vector spaces are precisely filtered colims of Smith spaces along injections.

Pf $\Rightarrow V$ M -complete.

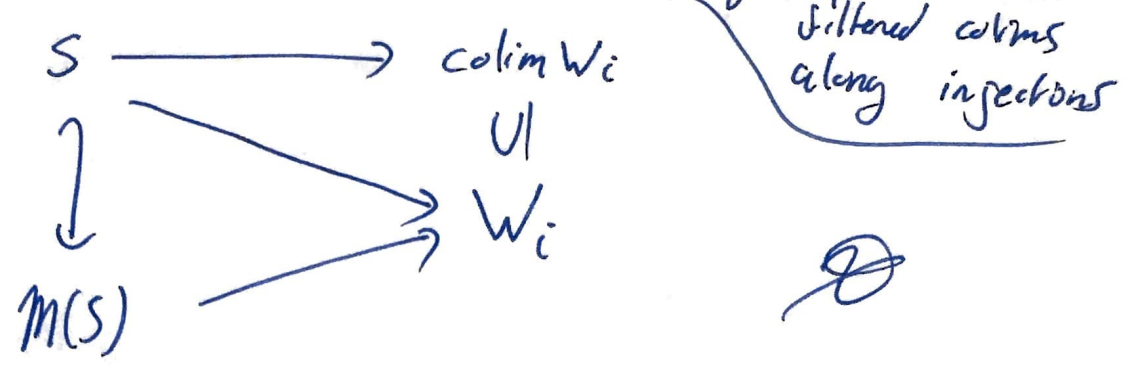
Take



Show M -Comp

\square

← Show $\text{colim } W_i$ is g_{sep} ~~is M -complete~~. ⑦



Can understand M -complete spaces in terms of Smith ones!

Nice Categorical Properties.

Prop (The Quasi-Separator)

The inclusion $\{g_{sep} \text{ cond}\} \xrightarrow{\quad} \{\text{cond}\}$ admits an adjoint.

Pf
 $X' = \coprod S_i \longrightarrow X$ S_i g_{sep} .

$R = X' \times_X X' \subseteq X' \times X'$ be induce eq. relation giving X .

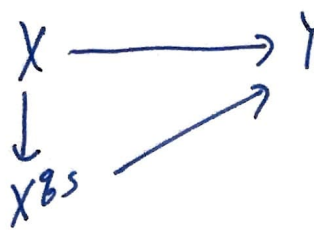
Suppose have $\begin{matrix} X & \longrightarrow & Y \\ \uparrow & \nearrow & \\ X' & & \end{matrix}$ w/ Y g_{sep} .

Get $X' \times_P X \subseteq X' \times X'$ g_{cpt} injection. Top lemma a closed subspace. Also contains R . Top lemma, makes

sense to let $\bar{R} =$ minimal closed eq relation containing R .

$X^{g_{sep}} = X' / \bar{R}$. Works by construction.

~~Hom(X, Y)~~



$\hookrightarrow R \subseteq X' \otimes Y'$ (8)
 $\frac{M}{R} \neq$

Theorem:

- 1) The category \mathcal{M} -complete admits kernels & cokernels ↗ sup?
- 2) The image & kernel of a map between \mathcal{M} -complete condensed sets is \mathcal{M} -complete.

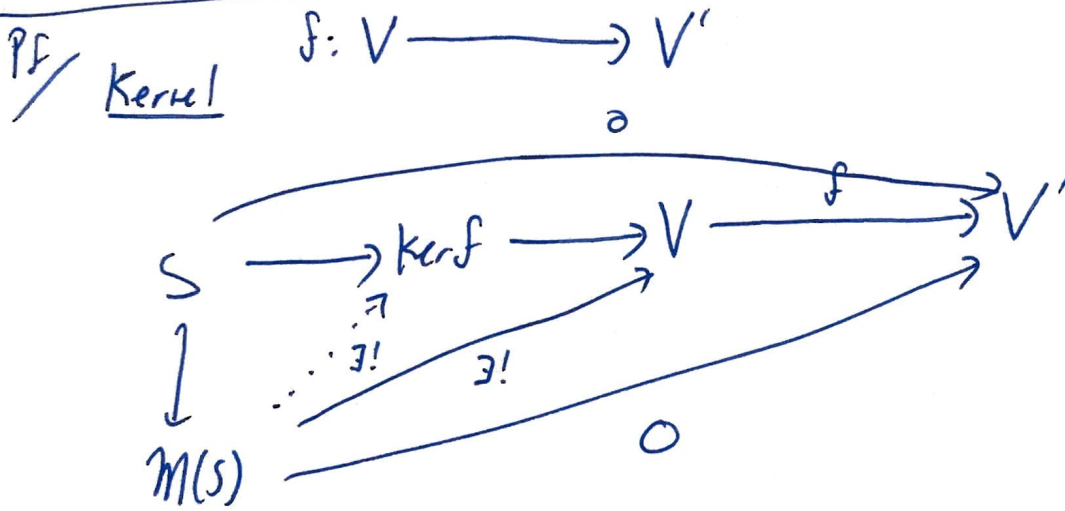
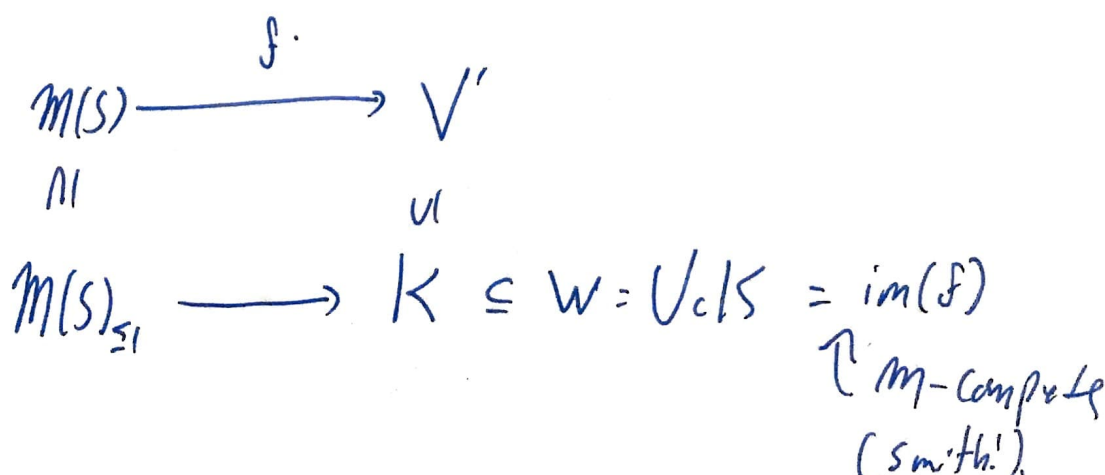


Image Noting V is colimit of $M(S)$, may assume $V = M(S)$.

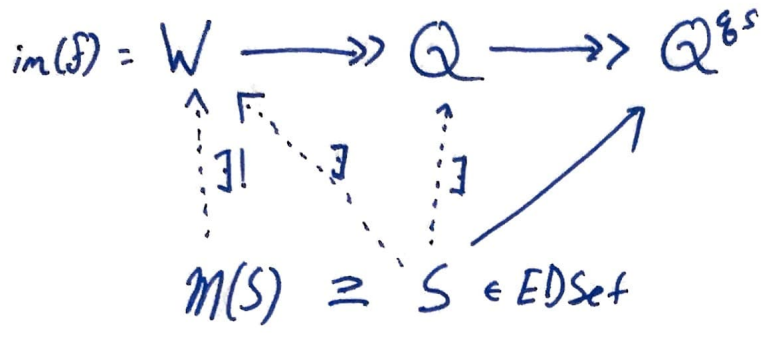


Cokernel

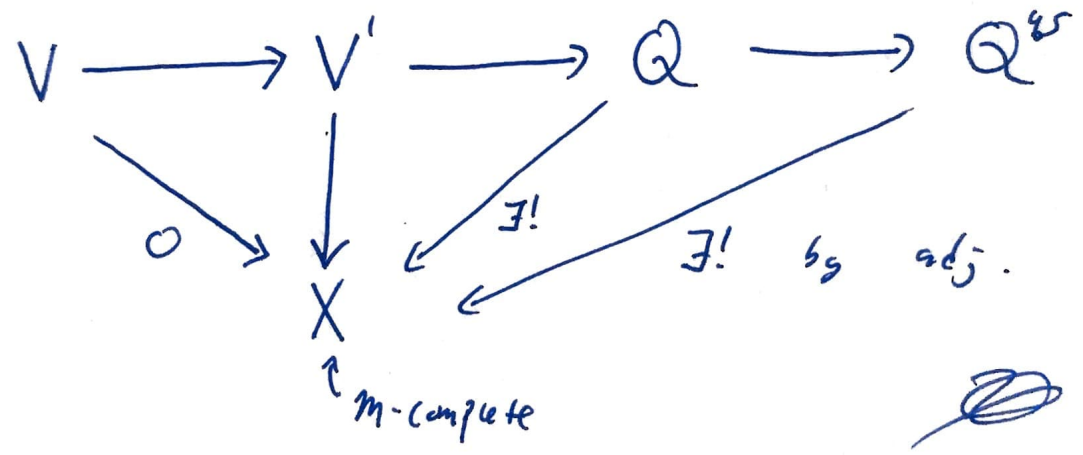
Let $Q = \text{coker}_{\text{cond}}(f)$

Show Q^{gs} works.

1) Show it is M -complete.



2) Show its cokernel.



Remarks

- 1) Set not closed under cokernels, but it does have cokernels.
- 2) May hope that M -complete $\forall \text{ of sup}$ may work, but this turns out to be false.

3) $M\text{-Compl} \xrightarrow{\hat{\quad}} \text{Cond} \mathbb{R}$ | $\text{per } \oplus \mathbb{R}[s_j] \rightarrow \oplus \mathbb{R}[s_i] \rightarrow V \rightarrow 0$
 $\hat{V} = \text{coker}(\oplus M(s_j) \rightarrow \oplus M(s_i))^{\text{gs}}$