

Toward Liquid Vector Spaces

①

Goals

- ① Revisit solid abelian groups from a philosophical perspective. Saw had nice categorical properties... but:
 - Why is it an analytic notion?
 - Why is it nonarchimedean? ~~not~~
 - How to tweak.
 - Take a measure-theoretic approach which will motivate the "liquid" picture.
- ② How do we implement this to get a nice ~~topo~~ category of condensed \mathbb{R} -vector spaces reflecting notion of completeness & other analytic notions).
- ③ What is a framework of analytic/algebraic geometry which contains both theories?
 - Analytic rings & solid/liquid modules.

② will probably take several weeks because it seems quite involved. I think it is ~~too~~ useful (at least to me) to first muse philosophically on ① & ③ to motivate the definitions & hard work ahead of us, in particular ~~on~~ having a framework in which we understand liquidity as really part of the same theory. So I'm going to start with this, ~~rather~~ focussing (rather imprecisely) on some examples & asking lots of questions. Discuss should proceed throughout.

Revisiting Solid Abelian Groups

(2)

Defⁿ

$S = \lim S_i$ be a profinite set, S_i finite.

$$\mathbb{Z}[S]^{\square} := \lim \mathbb{Z}[S_i]$$

If $M \in \text{Cond}(\text{Ab})$, we say M is solid if

$$\begin{array}{ccc} \mathbb{Z}[S] & \xrightarrow{f} & M \\ \downarrow & \nearrow \exists! & \\ \mathbb{Z}[S]^{\square} & & \end{array}$$

Examples

- The category of solid ~~mod~~ abelian groups forms an abelian category w/ limits, colimits, extensions.
- There is an inclusion $\text{Solid} \rightarrow \text{Cond}(\text{Ab})$ has a left adjoint $M \mapsto M^{\square}$ solidification.
- There is a tensor product $M \otimes^{\square} N := (M \otimes N)^{\square}$
- There are compact projective generators: $\mathbb{Z}[S]^{\square}$ $S \in \text{c.d. Set}$
 $\hat{\cong} \prod_I \mathbb{Z}$ I any set

Examples

• $\mathbb{Z}[S]^{\square}$

• $\mathbb{F}_p[[t]] = \text{coker}(\mathbb{Z}[[t]] \xrightarrow{-1-p} \mathbb{Z}[[t]])$

• $\mathbb{Z}/p^n\mathbb{Z} = \text{coker}(\mathbb{Z} \xrightarrow{-p^n} \mathbb{Z})$

• $\mathbb{Z}[[t]] \cong \prod_{\mathbb{N}} \mathbb{Z}$

• $\mathbb{Z}_p \cong \lim \mathbb{Z}/p^n\mathbb{Z}$

Remark $\mathbb{F}_p[[t]] \approx \mathbb{Z}_p \leftarrow$ topologically.

• $\mathbb{Q}_p = \varinjlim_{p^n} \mathbb{Z}_p \approx \bigcup \frac{1}{p^n} \mathbb{Z}_p$

• $\mathbb{F}_p((t))$. discrete

• $\mathbb{C}[[t]] \approx \mathbb{C} \otimes \mathbb{Z}[[t]]$, $\mathbb{C}((t))$. \leftarrow Nonarch rings appearing in A.G. / \mathbb{C}

Looks like just formally get many nonarchimedean rings.

Question: (I don't really see where analytic framework in this formal setting lives)

Can we directly show: $A =$ abelian group complete wrt nonarch abs value.

Show directly from $\mathbb{Z}[[t]]$ that it is solid?

Question:
Why isn't \mathbb{R} solid?

Measure Theoretic Perspective

Observation

S a finite set.

$$\begin{aligned} \mathbb{Z}[S] &\approx \text{Hom}(\overbrace{\text{Hom}(\mathbb{Z}[S], \mathbb{Z})}^{\text{continuous functions}}, \mathbb{Z}) \\ &\approx \text{Hom}(\text{Hom}(S, \mathbb{Z}), \mathbb{Z}) \\ &\approx \text{Hom}(\text{Cont}(S, \mathbb{Z}), \mathbb{Z}) =: \mathcal{M}(S) - \mathbb{Z}\text{-valued Radon measures.} \end{aligned}$$

Slogan = "dual to compactly supported continuous functions is Radon measures"

Therefore

(4)

If $S = \varprojlim S_i$ is profinite w/ S_i finite

$$\begin{aligned}\mathbb{Z}[S]^{\#} &= \lim \mathbb{Z}[S_i] \\ &= \lim \text{Hom}(\text{Cont}(S_i, \mathbb{Z}), \mathbb{Z}) \\ &= \text{Hom}(\text{colim} \text{Cont}(S_i, \mathbb{Z}), \mathbb{Z}) \\ &= \text{Hom}(\text{Cont}(\lim S_i, \mathbb{Z}), \mathbb{Z}) \\ &= \text{Hom}(\text{Cont}(S, \mathbb{Z}), \mathbb{Z}) = \mathcal{M}(S)\end{aligned}$$

$$\mathcal{M}(S) \longleftrightarrow \text{Hom}(\text{Cont}(S, \mathbb{Z}), \mathbb{Z})$$

Measures

$\mu: \left\{ \begin{array}{l} \text{some} \\ \text{subsets} \end{array} \right\} \text{ of } S \longrightarrow \mathbb{Z} \quad + \text{ properties.}$

\implies

$$\mu \longmapsto (g \longmapsto \int g d\mu)$$

S finite:

$$S = \{x_1, \dots, x_k\}$$

$$\mu(x_i) = \omega_i$$

$$\begin{aligned}\rightsquigarrow \int g d\mu &= \sum \mu(x_i) \cdot g(x_i) \\ &= \sum \omega_i \cdot g(x_i)\end{aligned}$$

Get profinite similarly. ("Measure theory works")

\Leftarrow Given $\phi: \text{Cont}(S, \mathbb{Z}) \longrightarrow \mathbb{Z}$.

$$\& I \subseteq S. \quad \mu_{\phi}(I) = ?$$

Reverse engineer this

(5)

Given $\phi: g \mapsto \int g d\mu$

We can recover μ .

Given $I \subseteq S$ define $\delta_I(x) = \begin{cases} 1 & x \in I \\ 0 & \text{else.} \end{cases}$

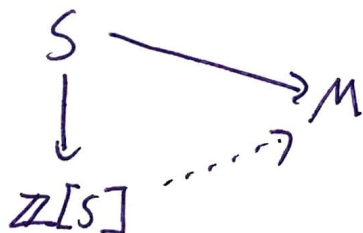
Then $\int_S \delta_I d\mu = \int_I 1 d\mu = \mu(I)$

$$\left(\int \delta_I = \sum_{x \in I} \mu(x) = \mu(I) \right)$$

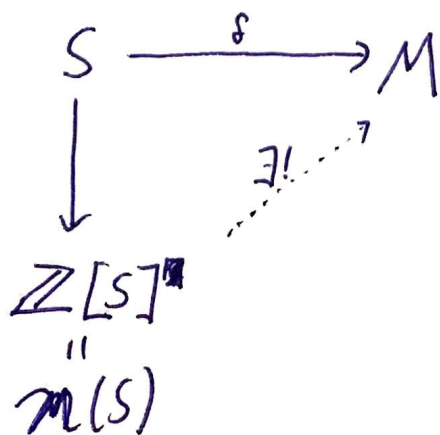
So: $\mu_\phi(I) = \phi(\delta_I)$

Reinterpret Solidity from this Perspective

Recall: $M \in \text{Cond}(Ab), S \in \text{pfSet}$



Solidity: $M \in \text{Solid}(Ab), S \in \text{PFSet}$



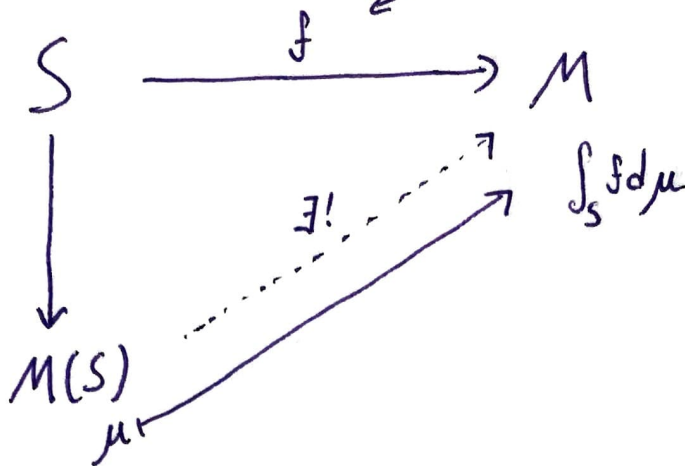
Continuous maps extend uniquely to functionals on measures.

Passing to underlying top spaces (or evaluating on T) says we can integrate continuous functions on \mathbb{Z} -valued Random Measures

On top

(evaluate @ pt)

cts map



6

Analyticity of Solidity:

~~Both~~ Solidity means you can integrate continuous functions on \mathbb{Z} -valued Radon Measures!

Example

Let $S = \mathbb{N} \cup \{\infty\}$

$= \lim S_i$

$S_i = \{0, 1, 2, \dots, i-1, \infty\}$

$$\begin{array}{ccc} S_{i+1} & \longrightarrow & S_i \\ i & \longrightarrow & \infty \end{array}$$

Topology on S

finite points are open.

$$i = \bigcap_{j \geq i} \pi_{i+1}^{-1}(i) \uparrow \text{discrete.}$$

$$\begin{array}{ccc} \pi_i : S & \longrightarrow & S_i \\ 0, 1, \dots, i-1 & \longrightarrow & 0, 1, \dots, i-1 \\ i, i+1, \dots, \infty & \longrightarrow & \infty \end{array}$$

Neighborhood basis for ∞

$\pi_i^{-1}(\infty) = \{i, i+1, i+2, \dots, \infty\}$

= sequences going to ∞ .

Let M be a topological space.

$$f: S \longrightarrow M$$

This is a sequence $m_0, m_1, m_2, \dots, m_\infty$ ($m_i = f(i)$)

And $\forall U \ni m_\infty$, $f^{-1}(U)$ is an open nhood of ∞ in S .

$$\text{So } f^{-1}(U) \supseteq \pi_i^{-1}(\infty) = \{i, i+1, \dots, \infty\}$$

$$\text{So } m_i, m_{i+1}, \dots, m_\infty \in U$$

$$\text{i.e. } \lim_{i \rightarrow \infty} m_i = m_\infty.$$

What is $\mathbb{Z}[S]^{\#} (*) = M(S)$?

$$N = \text{Cont.}(S, \mathbb{Z}) = \{n_0, n_1, n_2, \dots, n_N = n_{N+1} = \dots = n_\infty\}$$

A measure then is $\text{def } a_i = \mu(\{i\})$

$$\Delta \quad a = \mu(S)$$

$$\text{Given } \int n d\mu = \sum_{i=0}^{\infty} a_i (n_i - n_\infty) + a n_\infty \quad \begin{array}{l} i \geq N \\ n_i - n_\infty = 0 \end{array}$$

$$= \sum_{i=0}^N a_i (n_i - n_\infty) + a n_\infty$$

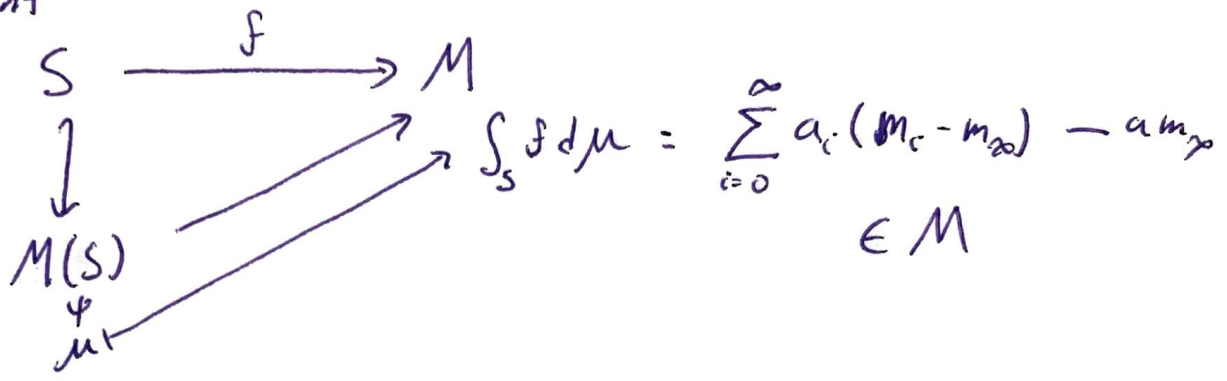
If $n_i \rightarrow 0$

$$\int n d\mu = \sum_{i=0}^N a_i n_i$$

M is solid & $S = \mathbb{N} \cup \{\infty\}$

Take m_i a sequence in M converging to m_∞

Same argument



For example: 1) Suppose $m_i \rightarrow 0$ (so $m_\infty = 0$)

2) $a_i = 1$ (trivial measure?)
(m_∞ will negative weight?)

Then M being solid $\implies \sum_{i=0}^{\infty} m_i \in M$

for any sequence $m_i \rightarrow 0$.

This is nonarchimedean!

\rightarrow M nonarch Banach space, this holds by strong Δ inequality

$\rightarrow M = \mathbb{R}$ fails spectacularly $(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots)$

This is evidence to me.

