Toward Liquid Vector Spaces

Goals

1. Revisit solid abelian groups from a philosophical perspective. Saw had nice categorical properties... but:
   - → Why is it an analytic notion?
   - → Why is it nonarchimedian? ~~[illegible]~~
   - → How to tweak.
   - → Take a measure-theoretic approach which will motivate the "liquid" picture.
2. How do we implement this to get a nice ~~[illegible]~~ category of condensed $\mathbb{R}$-vector spaces reflecting notion of completeness (& other analytic notions).
3. What is a framework of analytic/algebraic geometry which contains both theories?
   - → Analytic rings & solid/liquid modules.

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② will probably take several weeks because it seems quite involved. I think it is ~~[illegible]~~ useful (at least to me) to first muse philosophically on ① & ③ to motivate the definitions & hard work ahead of us, in particular ~~[illegible]~~ having a framework in which we understand liquidity as really part of the same theory. So I'm going to start with this, ~~rathe~~ focussing (rathe imprecisely) on some examples & asking lots of questions.

Discuss should proceed throughout.

Revisiting Solid Abelian Groups

Def^n

$S = \lim S_i$ be a profinite set, S_i finite.

$\mathbb{Z}[S]^\blacksquare := \lim \mathbb{Z}[S_i]$

If $M \in \mathrm{Cond}(Ab)$, we say M is solid if

$$\begin{tikzcd} \mathbb{Z}[S] \arrow[r, "f"] \arrow[d] & M \\ \mathbb{Z}[S]^\blacksquare \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Examples

- The category of solid abelian groups forms an abelian category w/ limits, colimits, extensions.
- The inclusion $\mathrm{Solid} \to \mathrm{Cond}(Ab)$ has a left adjoint $M \mapsto M^\blacksquare$ solidification.
- There is a tensor product $M \otimes^\blacksquare N := (M \otimes N)^\blacksquare$
- There are compact projective generators: $\mathbb{Z}[S]^\blacksquare$ $S \in$ e.d.Set $\hookrightarrow \prod_I \mathbb{Z}$ I any set

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Examples

- $\mathbb{Z}[S]^\blacksquare$
- $\mathbb{Z}[[t]] \simeq \prod_{\mathbb{N}} \mathbb{Z}$
- $\mathbb{F}_p[[t]] = \mathrm{coker}\left(\mathbb{Z}[[t]] \xrightarrow{\cdot p} \mathbb{Z}[[t]]\right)$
- $\mathbb{Z}/p^n\mathbb{Z} : \mathrm{coker}\left(\mathbb{Z} \xrightarrow{\cdot p^n} \mathbb{Z}\right)$
- $\mathbb{Z}_p \simeq \lim \mathbb{Z}/p^n\mathbb{Z}$.

Remark $\mathbb{F}_p[[t]] \simeq \mathbb{Z}_p$ $\leftarrow$ topologically.

- $\mathbb{Q}_p = \underset{p^n}{\mathrm{colim}} \, \mathbb{Z}_p \simeq \bigcup \frac{1}{p^n}\mathbb{Z}_p$

- $\mathbb{F}_p((t))$.

- $\mathbb{C}[[t]] \simeq \underset{\text{discrete}}{\mathbb{C}} \otimes^{\blacksquare} \mathbb{Z}[[t]]$, $\mathbb{C}((t))$. $\leftarrow$ Nonarch rings appearing in A.G. / $\mathbb{C}$.

Looks like just formally get many nonarchimedian rings.

Question: (I don't really see where analytic framework in this formal setting lives)

Can we directly show: A = abelian group complete wrt nonarch abs value.

Show directly from def^n that it is solid?

Question:

why isn't $\mathbb{R}$ solid?

Measure Theorik Perspective

Observation

S a finite set.

$$\mathbb{Z}[S] \simeq \mathcal{H}om(\overbrace{\mathcal{H}om(\mathbb{Z}[S], \mathbb{Z})}, \mathbb{Z})$$
$$\simeq \mathcal{H}om(\mathcal{H}om(S, \mathbb{Z}), \mathbb{Z})$$
$$\simeq \mathcal{H}om(\mathrm{Cont}(S, \mathbb{Z}), \mathbb{Z}). =: \mathcal{M}(S)$$ - $\mathbb{Z}$-valued Radon measures.

- Slogan = "dual to compactly supported continuous functions is Radon measures"

Therefore

If $S = \varprojlim S_i$ is profinite w/ S_i finite

$$
\begin{aligned}
\mathbb{Z}[S]^{\blacksquare} &= \lim \mathbb{Z}[S_i] \\
&= \lim \mathcal{H}om(\mathrm{Cont}(S_i, \mathbb{Z}), \mathbb{Z}) \\
&= \mathcal{H}om(\operatorname{colim} \mathrm{Cont}(S_i, \mathbb{Z}), \mathbb{Z}) \\
&= \mathcal{H}om(\mathrm{Cont}(\lim S_i, \mathbb{Z}), \mathbb{Z}) \\
&= \mathcal{H}om(\mathrm{Cont}(S, \mathbb{Z}) \mathbb{Z}) = \mathcal{M}(S)
\end{aligned}
$$

$$\mathcal{M}(S) \longleftrightarrow \mathrm{Hom}(\mathrm{Cont}(S, \mathbb{Z}), \mathbb{Z})$$

Measures

$\mu : \left\{ \begin{array}{c} \text{some subsets} \\ \text{of } S \end{array} \right\} \longrightarrow \mathbb{Z}$ + properties.

$\Longrightarrow$

$$\mu \longmapsto \left(g \longmapsto \int g \, d\mu \right)$$

S finite:

$S = \{x_1, \ldots, x_t\}$

$\mu(x_i) = \omega_i$

$$\rightsquigarrow \int g \, d\mu = \sum \mu(x_i) \cdot g(x_i) = \sum \omega_i \cdot g(x_i)$$

Get profinite similarly. ("Measure theory works")

$\Longleftarrow$ Given $\phi : \mathrm{Cont}(S, \mathbb{Z}) \longrightarrow \mathbb{Z}$.

& $I \subseteq S$. $\mu_\phi(I) = ?$

Reverse engineer this

Given $\phi : g \longmapsto \int g\, d\mu$

we can recover μ.

Given $I \subseteq S$ define $\delta_I(x) = \begin{cases} 1 & x \in I \\ 0 & \text{else.} \end{cases}$

Then $\int_S \delta_I \, d\mu = \int_I 1 \, d\mu = \mu(I)$

(finite $= \sum_{x \in I} \mu(x) = \mu(I)$)

So: $\mu_\phi(I) = \phi(\delta_I)$

Reinterpret Solidity from this Perspective

Recall: $M \in \mathrm{Cond}(\mathrm{Ab})$, $S \in \mathrm{pfSet}$

$$\begin{tikzcd} S \arrow[r] \arrow[d] & M \\ \mathbb{Z}[S] \arrow[ur, dashed] & \end{tikzcd}$$

Solidity: $M \in \mathrm{Solid}(\mathrm{Ab})$, $S \in \mathrm{PFSet}$

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d] & M \\ \mathbb{Z}[S]^{\blacksquare} \arrow[ur, dashed, "\exists!"] & \end{tikzcd}$$

$$\mathbb{Z}[S]^{\blacksquare} = \mathcal{M}(S)$$

Continuous maps extend uniquely to functionals on measures.

Passing to underlying top spaces (or evaluating on $\top$) says we can integrate continuous functions on $\mathbb{Z}$-valued Radon Measures

On top (evaluate @ pt)

cts map

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d] & M \\ M(S) \arrow[ur, dashed, "\exists!"] & \end{tikzcd}$$

$\mu \mapsto \int_S f \, d\mu$

Analyticity of Solidity:

~~Solid~~ Solidity means you can integrate continuous functions on $\mathbb{Z}$-valued Radon Measures!

Example

Let $S = \mathbb{N} \cup \{\infty\}$

$= \lim S_i$

$S_i = \{0, 1, 2, \dots, i-1, \infty\}$

$$\begin{aligned} S_{i+1} &\longrightarrow S_i \\ i &\longmapsto \infty \end{aligned}$$

Topology on S

finite points are open.

$i =$ ~~...~~ $\pi_{i+1}^{-1}(i)$

discrete.

$$\begin{aligned} \pi_i : S &\longrightarrow S_i \\ 0, 1, \dots, i-1 &\longmapsto 0, 1, \dots, i-1 \\ i, i+1, \dots \infty &\longmapsto \infty \end{aligned}$$

Nhood basis for ∞

$$\pi_i^{-1}(\infty) = \{i, i+1, i+2, \dots, \infty\}$$

= Sequences going to ∞.

Let M be a topological space.

$$f: S \longrightarrow M$$

This is a sequence $m_0, m_1, m_2, \dots, m_\infty$ $\quad (m_i = f(i))$

And $\forall$ $U \ni m_\infty$, $f^{-1}(U)$ is an open nhood of ∞ in S.

So $f^{-1}(U) \supseteq \pi_i^{-1}(\infty) = \{i, i+1, \dots, \infty\}$

So $m_i, m_{i+1}, \dots, m_\infty \in U$

i.e. $\lim_{i \to \infty} m_i = m_\infty$.

What is $\mathbb{Z}[S]^*(*) = M(S)$?

$n : \mathrm{Cont}(S, \mathbb{Z}) = n_0, n_1, n_2, \dots, n_N = n_{N+1} = \dots = n_\infty$

A measure then is det $a_i = \mu(\{i\})$

& $\quad a = \mu(S)$

Given $\int n \, d\mu = \sum_{i=0}^{\infty} a_i(n_i - n_\infty) + a n_\infty$ $\quad$ $i \geq N$ $\quad n_i - n_\infty = 0$

$$= \sum_{i=0}^{N} a_i(n_i - n_\infty) + a n_\infty$$

If $n_i \rightsquigarrow 0$

$$\int n \, d\mu = \sum_{i=0}^{N} a_i n_i$$

M is solid & $S = \mathbb{N} \cup \{\infty\}$

Take m_i a sequence in M converging to m_∞

Same argument

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d] & M \\ M(S) \arrow[ur] & \end{tikzcd}$$

$$\mu \mapsto \int_S f \, d\mu = \sum_{i=0}^{\infty} a_i (m_i - m_\infty) - a m_\infty \in M$$

For example: 1) Suppose $m_i \rightsquigarrow 0$ (so $m_\infty = 0$)

2) $a_i = 1$ (trivial measure?)
(m_∞ will negative weight?)

Then M being solid $\Longrightarrow \sum_{i=0}^{\infty} m_i \in M$

for any sequence $m_i \rightsquigarrow 0$.

This is nonarchimedean!

$\rightarrow$ M nonarch Banach space, this holds by strong Δ inequality

$\rightarrow$ $M = \mathbb{R}$, fails spectacularly $\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots\right)$

This is evidence to me.

3 What framework are we looking for?

Want: A a condensed ring.

A well behaved category of complete A-modules.

For $A = \mathbb{Z}$, these were solid ~~modules~~ abelian groups.

It was enough to specify free objects & test solidity against them.

$$(\mathbb{Z},\ S \longmapsto \mathbb{Z}[S]^{\blacksquare}) \leftarrow \text{Solid theory.}$$

For A, want a functor $S \longmapsto \widehat{A[S]}$

e.g.

$$S \longmapsto \mathbb{Z}[S]^{\blacksquare} \otimes^{\blacksquare} A$$

But again: $\mathbb{Z}[S]^{\blacksquare} \otimes^{\blacksquare} \mathbb{R} = 0$

So again theory fails for $\mathbb{R}$.

But like we saw, $\mathbb{Z}[S]^{\blacksquare} \leftarrow$ suitable space of $\mathbb{Z}$-measures on S.

Maybe $\widehat{A[S]} = \mathcal{M}(S, A) \leftarrow$ suitable space of A-measures on S.

Idea: Analytic Framework will be a pair

$$(A,\ S \longmapsto \widehat{A[S]} = \mathcal{M}(S, A))$$

$\uparrow$ complete free objects

"Completeness"
" integrating

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d] & M \\ A[S]^{\wedge} \arrow[ur, dashed] & \end{tikzcd}$$

$$\mu \mapsto \int f\,d\mu$$

Test objects are still profinite. Still leave nonarch setting.