[Coherent Duality]

1. Recall classical Serre Duality:

X : <u>proper smooth</u> / k, dim d.
$\omega_X := \det \Omega^1_{X/k}$.
Then $\cdot$ $\exists$ canonical $tr: H^d(X, \omega_X) \to k$
$\cdot$ $\forall$ coherent $\mathcal{F}$, $Ext^{d-i}(\mathcal{F}, \omega_X) \times H^i(X, \mathcal{F}) \to H^d(X, \omega_X) \xrightarrow{tr} k$ is perfect pairing.

<u>Question</u>: generalise to nonproper / nonsmooth / general base.

We will discuss the nonproper, ~~case~~ base = Spec R case, but still assume smoothness.
Main difficulty is in constructing $(f_! \dashv f^!)$, condensed theory allows us to "easily" do this.

2. First rewrite statement in derived terms:
- $tr: R\Gamma(X, \omega_X)[d] \to R$
- $R\mathcal{H}om_X(\mathcal{F}, \omega_X)[d] \to R\mathcal{H}om_R(R\Gamma(X, \mathcal{F}), R)$

(second map is induced via $LHS \to R\mathcal{H}om_R(R\Gamma(X, \mathcal{F}), R\Gamma(X, \omega_X))[d] \xrightarrow{tr} RHS$.)

Then Coherent Duality in our setting should take the following form:
To each (finite type$/_{\mathbb{Z}}$) scheme X, assign "quasicoherent solid sheaves" (studied in previous talks):
$D(X) := D(\mathcal{O}_{X,\blacksquare}) := D((\mathcal{O}_X^{ad}, \mathcal{O}_{X^{ad}}^+)_\blacksquare)$.
For affines $X = \operatorname{Spec} A$, this is just $D(A_\blacksquare) := D(A, A)_\blacksquare$.

This is functorial in X, for $f: X \to Y$, we have
f_*: forget, $f^*: - \otimes_{\mathcal{O}_Y} \mathcal{O}_X$:
$$\begin{tikzcd} D(X) \arrow[r, bend right, "f_*"'] \arrow[r, phantom, "\perp"] & D(Y) \arrow[l, bend right, "f^*"'] \end{tikzcd}$$.

Thm Let $f: X \to \operatorname{Spec} R$ be separated, <u>smooth</u>, finite type, dim d.
Let $\omega_X = \det \Omega^1_{X/R}$.

Then
- $\exists\ f_! : D(X) \to D(\operatorname{Spec} R) (= D(R_\blacksquare))$, which coincides with $f_* = R\Gamma(X, -)$ if f is proper.
- $\exists$ canonical $\mathrm{tr}: f_! \omega_X[d] \to R$
- $\forall\ \mathcal{F} \in D(X)$, the induced map $R\mathrm{Hom}_X(\mathcal{F}, \omega_X)[d] \to R\mathrm{Hom}_R(f_! \mathcal{F}, R)$ is an iso.

Moreover, $f_!$ preserves compact objects.

Think of ~~Thm~~ as having two parts:
① construction of $f_!$ ② explicit form of dualising sheaf.

3. Actually we can establish a full <u>Six Functors Formalism</u> for q.coh. solid sheaves, just as for
ℓ-adic sheaves; [redacted]; $\mathcal{D}$-modules...
In Grothendieck's words, "a complete mastery of a cohomology theory".

Let's summarise <u>Six Functors Formalism</u>:
① to each "space" X assign "coefficients" $D(X)$

② $D(X)$ has $-\otimes-$, $\underline{\mathrm{Hom}}(-,-)$, adjunction.

③ $f: X \to Y \rightsquigarrow f^* \dashv f_*$, $f^*: D(Y) \rightleftarrows D(X) : f_*$; ④ "good" $f: X \to Y \rightsquigarrow f_! \dashv f^!$, $f_!: D(X) \rightleftarrows D(Y) : f^!$

⑤ f proper $\rightsquigarrow$ $f_! = f_*$

⑥ f étale $\rightsquigarrow$ $f^! = f^*$

⑦ Duality: $\mathrm{Hom}_X(A, f^! B) \cong \mathrm{Hom}_Y(f_! A, B)$

⑧ Projection formula: $f_!(A \otimes_X f^! B) \cong (f_! A) \otimes_Y B$

⑨ Base change:
$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$
$g^* f_! \xrightarrow{\sim} f'_! g'^*$

(adjunction gives $f^! g_* \xrightarrow{\sim} g'_* f'^!$)

⑩ Relative purity: $f: X \to Y$ smooth, then
$$f^* B \otimes_X f^! 1_Y \xrightarrow{\sim} f^! B$$

(map induced by LHS $\xrightarrow{\text{unit}}$ $f^! f_!(f^* B \otimes f^! 1_Y) \xrightarrow{⑧} f^!(B \otimes f_! f^! 1_Y) \xrightarrow{\text{counit}}$ RHS)

Remarks:

- ⑩ means, to compute $f^!$(general), suffice to compute $f^! 1_Y$.

- $f_!$ is determined by ⑤, ⑥ (& f_*, f^*).

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & \overline{X} \arrow[d, "\overline{f}"] \\ & Y \end{tikzcd}$$

$f_! = \overline{f}_! j_! \overset{⑤}{=} \overline{f}_* j_! \overset{⑥}{=} \overline{f}_* \circ (\text{left adj. of } j^*)$.

- Key point in a six fun.-form. is construction of $(f_! \dashv f^!)$.

4. Let's look at our setting. For simplicity, we focus on affine case.

① (finite type/$\mathbb{Z}$) scheme $X \rightsquigarrow D(X) = D(\mathcal{O}_{X,\square})$.

② $- \otimes^L -$, $R\underline{Hom}(-,-)$ adjunction. ✓

③
$$\begin{tikzcd} D(X) \arrow[d, bend left, "f_*"] \\ D(Y) \arrow[u, bend left, "f^*"] \end{tikzcd}$$
$f^* \dashv f_*$ ✓

④ $f: X \to Y$ separated, finite type.
$$\begin{tikzcd} D(X) \arrow[d, bend right, "f_!"'] \\ D(Y) \arrow[u, bend right, "f^!"'] \end{tikzcd}$$
$f_! \dashv f^!$ (Lect. VIII)

(recall:
$$\begin{tikzcd} D(A_\square) \arrow[r, hook, "j_!"] \arrow[dr, "f_!"'] & D((A,R)_\square) \arrow[d, "f_*"] \\ & D(R_\square) \end{tikzcd}$$
in case $A = R[T]$, $j_!(-) = (-) \otimes_{(A,R)_\square} (A_\infty/A)[-1]$)

⑤ f proper $f_! = f_*$: ✓ because $\mathrm{Spa}(A,R)_\square \cong \mathrm{Spa}(R,R)$ if $\mathrm{Spec} A \to \mathrm{Spec} R$ is proper, by valuative criterion for properness.

⑥ ~~for open immersion, $D(A) \quad \mathrm{Spa}(A,A) \to \mathrm{Spa}(A,R)$ open imm. open + proper So $f_! = j_!$, so $j^* = j^* =$ right adj. to $j_! = j^!$. open imm. $\mathrm{Spa}(A,R)$~~ ¿?

⑦ formal from $(f_! \dashv f^!)$ and base change.

⑧ ✓ from Lect. VIII (or, as adjoint of ⑦)

⑨

⑩ ✓ from Lect. VIII (for f of fin. tor-dim)

$\overline{c_6}$

So far, we have obtained half of the Thm (for affine case).
The other half is to compute the dualising object.

5. Prop
$$\begin{tikzcd} \mathrm{Spec}B \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Spec}A \arrow[d, "f"] \\ \mathrm{Spec}S \arrow[r, "g"'] & \mathrm{Spec}R \end{tikzcd}$$

Assume:
- f is f.g., of fin. tor-dim
- g is flat (so f' is also of fin. tor-dim).

then ① $f^! \simeq f^* \otimes_{A} f^! R$

② $g'^* f^! \xrightarrow{\sim} f'^! g^*$

(map induced by $g'^* f^! \xrightarrow{\text{counit}} g'^* f^! g_* g^* \underset{\text{base change}}{\xrightarrow{\sim}} g'^* g'_* f'^! g^* \xrightarrow{\text{unit}} f'^! g^*$)

proof:

(a) closed imm.

We first prove ①. By Yoneda, it suffices to show:
$\forall N \in D(R_\blacksquare),\ L \in D(A_\blacksquare)$, $R\mathrm{Hom}_A(f^*N \otimes_{A_\blacksquare} f^!R, L) = R\mathrm{Hom}_A(f^!N, L)$.
Since f is of fin-tor-dim, $f_* = f_!$ preserves compact objects (Lec. VIII), this implies its right adj. $f^!$ in turn has a right adj., denote it by $f_\#$.

Then, $R\mathrm{Hom}_A(f^*N \otimes_{A_\blacksquare} f^!R, L) = R\mathrm{Hom}_A(f^*N, R\mathrm{Hom}_A(f^!R, L))$
$= R\mathrm{Hom}_A(N, f_* R\mathrm{Hom}_A(R, f_\# L))$
$= R\mathrm{Hom}_A(N, R\mathrm{Hom}_R(R, f_\# L))$
$= R\mathrm{Hom}_A(N, f_\# L) = R\mathrm{Hom}_A(f^!N, L)$. ✓

Thanks to Mark and Peter for helping with this proof!

To show ② first use adjunction $f_* = f_! \dashv f^!$ to find an explicit formula for $f^!$:
$f^!N = R\mathrm{Hom}_A(A, f^!N) = R\mathrm{Hom}_R(f_!A, N) = R\mathrm{Hom}_R(A, N)$.
So $f^!N = f^*N \otimes_{A_\blacksquare} R\mathrm{Hom}_R(A, R)$.
With this, ② is straightforward.

(b) $A = R[T]$. Here we already have ① by Lec VIII. To show ② we need an explicit formula for $f^!$. As above, $f^!N = R\mathrm{Hom}_R(f_!A, N)$.
So $f^!R = R\mathrm{Hom}_R(f_!A, R)$.
Now recall from Lec VIII: $f_!A = A \otimes_{(A,R)_\blacksquare} (A_\infty/A)[-1]$. Since A, A_∞/A are discrete hence already solid, tensor can be taken as A-mod.,
So actually $f_!A = (A_\infty/A)[-1]$.
So $f^!R = R\mathrm{Hom}_R(A_\infty/A, R)[1] = A[1]$.

With this, ② is straightforward. $\square$

[As R-Mod., $A_\infty/A \cong \prod R$
then use fact
$R\mathrm{Hom}_R(\prod R, R) \cong \bigoplus R\mathrm{Hom}_R(R, R) \cong R[T]$.]

6. Now we compute $f^!R$ for closed imm. and smooth map.

a) Closed immersion. $f : \mathrm{Spec}\,A \to \mathrm{Spec}\,R$. $A = R/I$.

Assume I is generated by a regular sequence. (algebraic ~~[illegible]~~ version of locally complete intersection.)

From 5., $f^!R = R\mathrm{Hom}_R(R/I, R)$. The following computation is classical. (e.g. Hartshorne III.7)

Choose a generating reg. seq. $(f_1, \dots, f_c)$ (c = codim of $\mathrm{Spec}\,A$ in $\mathrm{Spec}\,R$)

Koszul complex

$$0 \to \underset{-c}{\wedge^c K_1} \to \cdots \to \underset{-2}{\wedge^2 K_1} \to \underset{-1}{K_1} \to \underset{0}{R} \to 0$$

is a free resolution of R/I; here, K_1 = free R-mod. with generators $[f_1], \dots, [f_c]$.

$K_{-j} \to K_{1-j}$:

$$[f_{i_1}] \wedge \cdots \wedge [f_{i_j}] \longmapsto \sum_{l=1}^{j} (-1)^{l+1} f_{i_l} [f_{i_1}] \wedge \cdots \wedge \widehat{[f_{i_l}]} \wedge \cdots \wedge [f_{i_j}]$$

So $R\mathrm{Hom}_R(R/I, R) = R\mathrm{Hom}_R(K_\bullet, R)$

$$\cong R/I[-c]$$

(induced by evaluating on $[f_1] \wedge \cdots \wedge [f_c] \in \wedge^c K_1$)

This iso. depends on choice of $\{f_i\}$.

changing $\{f_i\} \to \{\sum_j c_{ij} f_j\}$ changes the iso. by $\times \det(c_{ij})$.

Note $\det(I/I^2)^\vee \cong R/I$ via $f_1 \wedge \cdots \wedge f_c \mapsto 1$

so the iso. $R\mathrm{Hom}_R(R/I, R) \xrightarrow{\sim} \det(I/I^2)^*[-c]$ is independent of choice.

<u>Upshot</u> $f^!R = R\mathrm{Hom}_R(R/I, R) \cong \det(I/I^2)^*[-c]$.

b) Smooth map

$f: \operatorname{Spec} A \to \operatorname{Spec} R$ smooth, dim $= d$.

We use the following trick & a) to compute $f^! R$.

Consider

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[dr, hook, "\Delta"] & & \\
& \operatorname{Spec}(A \otimes_R A) \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\lrcorner"] & \operatorname{Spec} A \arrow[d, "f"] \\
& \operatorname{Spec} A \arrow[r, "f"'] & \operatorname{Spec} R
\end{tikzcd}$$

$$\begin{aligned}
f^! R &= \Delta^! p_1^! f^! R & & p_1 \circ \Delta \simeq \mathrm{id} \\
&= \Delta^! \left(p_1^* f^! R \overset{L}{\underset{(A\otimes_R A)_\square}{\otimes}} p_1^! A \right) & & p_1^! = p_1^* \otimes p_1^! A \\
&= \Delta^! \left(p_1^* f^! R \overset{L}{\underset{(A\otimes_R A)_\square}{\otimes}} p_2^* f^! R \right) & & A = f^* R, \quad p_1^! f^* = p_2^* f^! \\
&= \Delta^* \left(p_1^* f^! R \overset{L}{\underset{(A\otimes_R A)_\square}{\otimes}} p_2^* f^! R \right) \overset{L}{\underset{A_\square}{\otimes}} \Delta^! (A \otimes_R A) & & \Delta^! = \Delta^* \otimes \Delta^! (A \otimes_R A) \\
&= f^! R \overset{L}{\underset{A_\square}{\otimes}} f^! R \overset{L}{\underset{A_\square}{\otimes}} (\det \Omega^1_{A/R})^* [-d] & & \text{use a), and } I/I^2 \cong \Omega^1_{A/R}
\end{aligned}$$

Since from computation in Lec. VIII, we know $f^! R$ is invertible, the above gives

$$f^! R \cong \det \Omega^1_{A/R} [d] \quad \left(=: \omega_{A/R} \right)$$

The Thm is proved by now.