10/29/21

Recall: Last time, we were trying to show that

$$D(\mathrm{Solid}) \hookrightarrow D(\mathrm{Cond}(\mathrm{Ab}))$$

is fully faithful. Let's try again.

$D(\mathrm{Solid})$ is generated by $\{\mathbb{Z}[S]^{\blacksquare}\}_{S \text{ extremely disconnected}}$ under ∞-colimits:

- geometric realizations give all bounded above complexes.
- filtered colimits give all complexes.

Thus, we need to show

$$\underset{D(\mathrm{Solid})}{B\mathrm{Hom}}(\mathbb{Z}[S]^{\blacksquare}, \mathcal{Y}) = \underset{D(\mathrm{Cond}(\mathrm{Ab}))}{B\mathrm{Hom}}(\mathbb{Z}[S]^{\blacksquare}, \mathcal{Y})$$

for S extremally disconnected. By writing

$$\mathcal{Y} = \lim_i \tau^{\geq i} \mathcal{Y},$$

we may assume $\mathcal{Y} \in D^{>-\infty}$.

We have a triangle

$$\tau^{\leq N} y \to y \to \tau^{\geq N+1} y \to . \quad (*)$$

Note that

$$R\mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare}, \tau^{\geq N+1} y) = 0$$

for $N \gg 0$, in either category. Indeed, $\mathbb{Z}[S] \in \mathrm{Solid} \subseteq \mathrm{Cond}$ $\subseteq D^{\leq 0}$. Furthermore, $D^{\leq 0}$ is left $\perp$ to $D^{\geq n}$ for $n > 0$. Thus, we may assume that $y \in D^{<\infty}$.

Therefore, y is bounded. By using $(*)$ and induction, we may assume $y \in \mathrm{Solid}$. Then we must show

$$\mathrm{Ext}^i_{\mathrm{Solid}}(\mathbb{Z}[S]^{\blacksquare}, y) = \mathrm{Ext}^i_{\mathrm{Cond}}(\mathbb{Z}[S]^{\blacksquare}, y)$$

$$= \mathrm{Ext}^i_{\mathrm{Cond}}(\mathbb{Z}[S], y)$$

derived solidity of y.

For $i > 0$, both sides vanish. For $i = 0$, it is clear.

$\square$

Prop: $X \in D(\mathrm{Cond})$ is solid iff $H^i(X)$ is solid $\forall i$.

Proof: Let $D_{solid} = \{X \in D(\mathrm{Cond}) \mid X \text{ is solid}\}$
and
$$D'_{solid} = \{X \in D(\mathrm{Cond}) \mid H^i(X) \in \mathrm{Solid}\ \forall i\}.$$

By definition, D'_{solid} is generated under ∞-colimits by the $\mathbb{Z}[S]^{\blacksquare}$'s.

D_{solid} is closed under colimits.

Thus, it suffices to show $\mathbb{Z}[S]^{\blacksquare} \in D_{solid}$. We already know this. $\square$

Recall that we must still verify the "main lemma":

Lemma: Let $f : Y \to Z$ be a morphism of direct sums of $\mathbb{Z}[S]^{\blacksquare}$'s. Then, $K = \ker f$ K is derived solid.

⚠ We used this lemma to prove the theorem.

<u>Lemma 1</u>: Let C be a bounded above complex

$$\cdots \to C_2 \to C_1 \to 0$$

such that each C_i is a sum of $\mathbb{Z}[S]^\blacksquare$'s. Then, C is (derived) solid.

<u>Lemma 1 $\Rightarrow$ Main Lemma:</u>

Let $f: \mathcal{Y} \to \mathcal{Z}$ be as in the main lemma. Then choose a resolution

$$\cdots \to B_1 \to B_0 \to K \to 0$$

where each B_i is a sum of $\mathbb{Z}[S]$'s; say,

$$B_i = \bigoplus_{j \in J_i} \mathbb{Z}[S_{ij}].$$

Let

$$C_i = \bigoplus_{j \in J_i} \mathbb{Z}[S_{ij}]^\blacksquare.$$

The C_i form a complex C. There is a morphism

$$B \longrightarrow C.$$

Applying Lemma 1 to $\mathcal{Y}$ yields

$$R\mathrm{Hom}(B, \mathcal{Y}) = R\mathrm{Hom}(C, \mathcal{Y}).$$

Similarly,

$$R\mathrm{Hom}(B, \mathbb{Z}) = R\mathrm{Hom}(C, \mathbb{Z}).$$

Therefore, $B \to K$ extends to a map $C \to K$.
But $B \to K$ is a quasi-iso, so we obtain a retraction

$$\begin{tikzcd} B \arrow[r] & C \arrow[l, bend right] \end{tikzcd}$$

Thus,

$$C \simeq B \oplus A .$$

So, for T profinite,

$$\begin{tikzcd}
R\mathrm{Hom}(\mathbb{Z}[T], C) \arrow[r, "\sim", "\text{Lemma 1}"'] \arrow[d, equal] & R\mathrm{Hom}(\mathbb{Z}[T]^{\blacksquare}, C) \arrow[d, equal] \\
\underbrace{R\mathrm{Hom}(\mathbb{Z}[T], B)} \oplus R\mathrm{Hom}(\mathbb{Z}[T], A) \arrow[r, "\sim"] & \underbrace{R\mathrm{Hom}(\mathbb{Z}[T]^{\blacksquare}, B)} \oplus R\mathrm{Hom}(\mathbb{Z}[T]^{\blacksquare}, C)
\end{tikzcd}$$

These factors match up.

$\square$

We will actually prove Lemma 1 only assuming that each C_i is a sum of $\prod_I \mathbb{Z}$'s (every $\mathbb{Z}[S]^{\blacksquare}$ has this form).

We will reduce Lemma 1 to Lemma 2, we'll state soon. For S profinite, recall that

$$\mathbb{Z}[S]^{\blacksquare} = \mathcal{M}(S,\mathbb{Z}) := \underline{\mathrm{Hom}}(C(S,\mathbb{Z}),\mathbb{Z}) \simeq \prod_I \mathbb{Z}.$$

We also define

$$\mathcal{M}(S,\mathbb{R}) := \underline{\mathrm{Hom}}(C(S,\mathbb{Z}),\mathbb{R}) \simeq \prod_I \mathbb{R}$$

$$\mathcal{M}(S,\mathbb{R}/\mathbb{Z}) := \underline{\mathrm{Hom}}(C(S,\mathbb{Z}),\mathbb{R}/\mathbb{Z}) \simeq \prod_I \mathbb{R}/\mathbb{Z}.$$

We have an exact sequence

$$0 \to \mathcal{M}(S,\mathbb{Z}) \to \mathcal{M}(S,\mathbb{R}) \to \mathcal{M}(S,\mathbb{R}/\mathbb{Z}) \to 0. \quad (*)$$

<u>Lemma 2</u>: Let C be a complex

$$\cdots \to C_2 \to C_1 \to 0$$

with each $C_i \simeq \bigoplus \prod \mathbb{Z}$. For any profinite sets S, S',

$$\underline{\mathrm{RHom}}(\mathcal{M}(S,\mathbb{R}/\mathbb{Z}), C)(S') = R\Gamma(S\times S', C)[-1].$$

<u>Lemma 2 $\Rightarrow$ Lemma 1</u>:

Take $S = *$:

$$\underline{\mathrm{RHom}}(\mathbb{R}/\mathbb{Z}, C)(S') = R\Gamma(S', C)[-1]$$
$$= \underline{\mathrm{RHom}}(\mathbb{Z}[1], C).$$

The SES

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$$

gives

$$\underline{R\mathrm{Hom}}(\mathbb{R}, C) = 0.$$

Thus,

$$\underline{R\mathrm{Hom}}(\mathcal{M}(S, \mathbb{R}), C) = \underline{R\mathrm{Hom}}_{\mathbb{R}}(\mathcal{M}(S, \mathbb{R}), \underline{R\mathrm{Hom}}(\mathbb{R}, C))$$
$$= 0.$$

Applying (*) gives

$$\underline{R\mathrm{Hom}}(\mathcal{M}(S, \mathbb{Z}), C) \simeq \underline{R\mathrm{Hom}}(\mathcal{M}(S, \mathbb{R}/\mathbb{Z}), C)[1].$$

Evaluate on S':

$$\underline{R\mathrm{Hom}}(\mathbb{Z}[S]^{\blacksquare}, C)(S') \simeq \underline{R\mathrm{Hom}}(\mathcal{M}(S, \mathbb{R}/\mathbb{Z}), C)(S')[1]$$
$$\underset{\text{Lemma 2}}{\simeq} R\Gamma(S \times S', C).$$

Take $S' = *$ to get Lemma 1. $\square$

Now we show Lemma 2.

Step 1: Let $C = M[0]$. Then

$$\underline{RHom}\Big(\mathcal{M}(S, \mathbb{R}/\mathbb{Z}), \bigoplus_{i \in I} \prod_{j \in J_i} \mathbb{Z}\Big)(S')$$

$$\cong \bigoplus_i \underline{RHom}\Big(\mathcal{M}(S, \mathbb{R}/\mathbb{Z}), \prod_{j \in J_i} \mathbb{Z}\Big)(S')$$

pseudo coherence of $\mathcal{M}(S, \mathbb{R}/\mathbb{Z})$ i.e. $RHom(\cdot, -)$ commutes w/ filtered colimits.

$$= \bigoplus_i \prod_{j \in J_i} \underline{RHom}(\mathcal{M}(S, \mathbb{R}/\mathbb{Z}), \mathbb{Z})(S')$$

$\overbrace{\prod \mathbb{R}/\mathbb{Z}}^{\text{cpct by Tychonoff}}$

$$= \bigoplus_i \prod_{j \in J_i} C(S \times S', \mathbb{Z})[-1] \quad (\text{from Tong's lectures})$$

$$= \bigoplus_i R\Gamma\Big(S \times S', \prod_{j \in J_i} \mathbb{Z}\Big)[-1]$$

$$= R\Gamma\Big(S \times S', \bigoplus_i \prod_{j \in J_i} \mathbb{Z}\Big)[-1]$$

$$= R\Gamma(S \times S', M[0])[-1].$$

Step 2: The case of bounded C follows by induction.

Step 3: It suffices to show that

$$\underline{RHom}(\mathcal{M}(S, \mathbb{R}/\mathbb{Z}), C) \text{ and } R\Gamma(S, C)$$

are in degrees ≤ 1, for S profinite. Indeed, let

$$C_{\leq i} \to C \to C_{\geq i+1} \to .$$

By taking $i \gg 0$ and invoking the bounded case, the lemma follows.

<u>Step 4</u>: Write $C_i = \bigoplus_i \prod_{j \in J_i} \mathbb{Z}$. Let

$$(C_{\mathbb{R}})_i = \bigoplus_i \prod_{j \in J_i} \mathbb{R}.$$

$C_{\mathbb{R}}$ is a complex. To see this, we show

$$\mathrm{Hom}(C_{i+1}, C_i) = \mathrm{Hom}(C_{\mathbb{R}, i+1}, C_{\mathbb{R}, i}).$$

We can assume $C_{i+1} = \prod_j \mathbb{Z}$, $C_{\mathbb{R}, i+1} = \prod_j \mathbb{R}$.
Then it STS that

$$R\mathrm{Hom}\Big(\prod_j \mathbb{R}/\mathbb{Z}, C_{\mathbb{R}, i}\Big) = 0.$$

By pseudocoherence of $\prod_j \mathbb{R}/\mathbb{Z}$, can assume

$$C_{\mathbb{R}, i} = \prod_k \mathbb{R},$$

in which case Tony stated the result.

We also define

$$(C_{\mathbb{R}/\mathbb{Z}})_i = \bigoplus_k \prod_j \mathbb{R}/\mathbb{Z} .$$

We have a SES

$$0 \to C_{\mathbb{Z}} \to C_{\mathbb{R}} \to C_{\mathbb{R}/\mathbb{Z}} \to 0 .$$

It therefore STS that

$$R\Gamma(\mathcal{M}(S, \mathbb{R}/\mathbb{Z}), C_{\mathbb{R}}), R\Gamma(S, C_{\mathbb{R}}),$$

$$R\Gamma(\mathcal{M}(S, \mathbb{R}/\mathbb{Z}), C_{\mathbb{R}/\mathbb{Z}}), R\Gamma(S, C_{\mathbb{R}/\mathbb{Z}})$$

are in degrees ≤ 0.

By passing to limits, we can prove the result for $\tau^{\leq i} C_{\mathbb{R}}$ and $\tau^{\leq i} C_{\mathbb{R}/\mathbb{Z}}$. It is enough to prove the result for the terms of these complexes. The only term for which the result doesn't follow from Step 2 is $\ker d_i$. So we must show that

$\underline{R\mathrm{Hom}}(\mathcal{M}(S, \mathbb{R}/\mathbb{Z}), \ker d_{\mathbb{R},i})$, $R\Gamma(S, \ker d_{\mathbb{R},i})$

$\underline{R\mathrm{Hom}}(\mathcal{M}(S, \mathbb{R}/\mathbb{Z}), \ker d_{\mathbb{R}/\mathbb{Z},i})$, $R\Gamma(S, \ker d_{\mathbb{R}/\mathbb{Z},i})$

are in degrees ≤ 1. Write

$$C_i = \bigoplus_{j \in J_i} \prod_{k \in K_{i,j}} \mathbb{Z}.$$

We can assume that $|J_i| = 1$. So, the differential is a map

$$d_{\mathbb{R}/\mathbb{Z},i} : \prod_{K_{i+1}} \mathbb{R}/\mathbb{Z} \to \prod_{K_i} \mathbb{R}/\mathbb{Z}.$$

Thus, $A = \ker d_{\mathbb{R}/\mathbb{Z},i}$ is compact. So,

$$\underline{R\mathrm{Hom}}(\mathcal{M}(S, \mathbb{R}/\mathbb{Z}), A)$$

is in degrees ≤ 1. Also, $R\Gamma(S, A)$ is in degrees ≤ 1 (we may resolve A in terms of $\mathbb{R}/\mathbb{Z}$'s).

The map

$$d_i : \prod_{K_{i+1}} \mathbb{Z} \longrightarrow \prod_{K_i} \mathbb{Z}$$

is dual to a map

$$\partial_i : \bigoplus_{K_i} \mathbb{Z} \to \bigoplus_{K_{i+1}} \mathbb{Z}.$$

The corresponding map $\partial_{i,\mathbb{R}} : \bigoplus_{K_i} \mathbb{R} \to \bigoplus_{K_{i+1}} \mathbb{R}$ has cokernel of the form $\bigoplus_{L} \mathbb{R}$ (by linear algebra).

Thus, $\ker d_i = \prod_{L} \mathbb{R}$. So,

$$\underline{R\mathrm{Hom}}(\underset{\substack{\wr\\ \prod \mathbb{R}/\mathbb{Z}}}{\mathcal{M}(S, \mathbb{R}/\mathbb{Z})}, \underset{\substack{\parallel\\ \prod \mathbb{R}}}{\ker d_{\mathbb{R},i}}) = 0.$$

OTOH, $R\Gamma(S, \ker d_{\mathbb{R},i})$ is in degree 0 by the vanishing of the cohomology of profinite sets with $\mathbb{R}$-coefficients.

$\square$

<u>Cor</u>: (i) The compact projectives in Solid all have the form

$$\prod_I \mathbb{Z}.$$

(ii) $D(\text{Solid})$ is compactly generated. The full subcategory $D(\text{Solid})^\omega$ of compact objects consists of all bounded complexes with terms

$$\prod_I \mathbb{Z}.$$

There is an equivalence

$$D(\text{Solid})^\omega \simeq D(\text{Ab})^{op}$$

$$R\underline{\text{Hom}}(C, \mathbb{Z}) \longleftarrow C$$

(iii) $\mathbb{R}^{L\blacksquare} = 0$.

(iv) If $C \in D(\text{Solid})$, S profinite,

$$R\underline{\text{Hom}}(\mathbb{Z}[S], C) = R\underline{\text{Hom}}(\mathbb{Z}[S]^\blacksquare, C).$$

I.e. this holds with internal hom!

__Proof__: (iii) Let $C \in D(\mathrm{Solid})$. Then,

$$R\underline{\mathrm{Hom}}(\mathbb{R}, C) = 0.$$

Indeed, we can reduce to $C = \bigoplus \prod \mathbb{Z}$.

Then it follows from the previous proof.

Thus, by Yoneda, $\mathbb{R}^{L\blacksquare} = 0$.

Solid Tensor Product

Thm: Solid has a symmetric monoidal product

$$- \overset{\blacksquare}{\otimes} -$$

Such that

$$\begin{aligned} \text{Cond} &\longrightarrow \text{Solid} \\ M &\longmapsto M^{\blacksquare} \end{aligned}$$

is symmetric monoidal.

Proof: We define, for $M, N \in$ Solid,

$$M \overset{\blacksquare}{\otimes} N := (M \otimes N)^{\blacksquare}$$

This is clearly a symmetric monoidal product. To check that $M \mapsto M^{\blacksquare}$ is symmetric monoidal, we must show that

$$(M \otimes N)^{\blacksquare} \simeq (M^{\blacksquare} \otimes N^{\blacksquare})^{\blacksquare}.$$

It STS that

$$(M \otimes N)^{\blacksquare} = (M^{\blacksquare} \otimes N)^{\blacksquare}.$$

All functors in question commute with colimits, so we can assume

$$M = \mathbb{Z}[S] \qquad N = \mathbb{Z}[T].$$

We must check that

$$\mathbb{Z}[S\times T]^{■} \xrightarrow{\sim} (\mathbb{Z}[S]^{■} \otimes \mathbb{Z}[T])^{■}.$$

Thus, we must show that $\forall\, A \in \mathrm{Solid}$,

$$\begin{tikzcd}
\mathrm{Hom}(\mathbb{Z}[S\times T]^{■}, A) \arrow[r, "\simeq"] \arrow[d, equal] & \mathrm{Hom}(\mathbb{Z}[S]^{■} \otimes \mathbb{Z}[T], A) \arrow[d, equal] \\
\mathrm{Hom}(\mathbb{Z}[S\times T], A) \arrow[dr, equal] & \underline{\mathrm{Hom}}(\mathbb{Z}[S]^{■}, A)(T) \arrow[d, "\wr", "\text{Cor.}"'] \\
& \underline{\mathrm{Hom}}(\mathbb{Z}[S], A)(T)
\end{tikzcd}$$

$\square$

As a corollary, we get a derived $\otimes$:

$$- \overset{L■}{\otimes} - : D(\mathrm{Solid}) \times D(\mathrm{Solid}) \longrightarrow D(\mathrm{Solid}),$$

and $D(\mathrm{Cond}) \xrightarrow{(-)^{L■}} D(\mathrm{Solid})$ is symmetric monoidal.

Example: $M = \prod_I \mathbb{Z}$ $N = \prod_J \mathbb{Z}$. Then

$$M \otimes^{L\blacksquare} N = \prod_{I \times J} \mathbb{Z}.$$

Example: Let $p \neq \ell$ be prime. Then

$$① \quad \mathbb{Z}_p \otimes^{L\blacksquare} \mathbb{R} = 0$$

$$⑤ \quad \mathbb{Z}_p \otimes^{L\blacksquare} \mathbb{Z}_\ell = 0$$

$$④ \quad \mathbb{Z}_p \otimes^{L\blacksquare} \mathbb{Z}_p = \mathbb{Z}_p$$

$$③ \quad \mathbb{Z}_p \otimes^{L\blacksquare} \mathbb{Z}[[T]] = \mathbb{Z}_p[[T]].$$

$$② \quad \mathbb{Z}[[u]] \otimes^{L\blacksquare} \mathbb{Z}[[T]] = \mathbb{Z}[[u,T]].$$

Pf: ① $\mathbb{Z}_p \otimes^{L\blacksquare} \mathbb{R} = \mathbb{Z}_p^{\blacksquare} \otimes^{L\blacksquare} \underbrace{\mathbb{R}^{\blacksquare}}_{=0} = 0$

② Write $\mathbb{Z}[[u]] = \prod_{i \in \mathbb{N}} \mathbb{Z}$. Then

$$\mathbb{Z}[[u]] \otimes^{L\blacksquare} \mathbb{Z}[[T]] = \prod_{i \in \mathbb{N}} \mathbb{Z} \otimes^{L\blacksquare} \prod_{j \in \mathbb{N}} \mathbb{Z}$$

$$= \prod_{i \in \mathbb{N} \times \mathbb{N}} \mathbb{Z}$$

$$= \mathbb{Z}[[T,u]].$$

③ Mod out by $(u-p)$ and use right exactness.

(4) Obtain from (3) as above.

(5) obtain from (3) as above; note that

$$\mathbb{Z}[[T]] \overset{L\blacksquare}{\otimes} \mathbb{Z}_p \xrightarrow{\times (T-p)} \mathbb{Z}[[T]] \overset{L\blacksquare}{\otimes} \mathbb{Z}_p$$

is an iso., so $\mathbb{Z}_p \overset{L\blacksquare}{\otimes} \mathbb{Z}_p \simeq 0$. $\square$

<u>Example</u>: X be a CW complex, viewed as a condensed set. Then

$$\mathbb{Z}[X]^{L\blacksquare} \simeq C_\bullet(X).$$

<u>Proof</u>: By passing to colimits, we may assume that X is a finite CW complex, hence compact + Hausdorff.

...