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Solid abelian groups

§ 1 Definition of solid abelian groups

Def. Let S be a profinite set. We define

$$\mathbb{Z}[S]^{\blacksquare} = \varprojlim \mathbb{Z}[S_i] \qquad (\text{for } S = \varprojlim S_i)$$

Recall: The forgetful functor $\mathrm{Cond}(\mathrm{Ab}) \to \mathrm{Cond}(\mathrm{Set})$ has a left adjoint

$$\begin{array}{ccc} \mathrm{Cond}(\mathrm{Set}) & \longrightarrow & \mathrm{Cond}(\mathrm{Ab}) \\ \Psi & & \Psi \\ T & \longmapsto & \mathbb{Z}[T] \end{array}$$

described explicitly by

$$\mathbb{Z}[T] = (U \mapsto \mathbb{Z}[\mathrm{Maps}(U, T)])^{sh}.$$

For $S \in \mathrm{FinSet}$,

$$\mathbb{Z}[S] \simeq \underbrace{\mathbb{Z}[S]}$$

group algebra of S, regarded as a condensed abelian group.

Rmk: For each i, we have a map $S \to S_i$, which induces

$$\mathbb{Z}[S] \longrightarrow \mathbb{Z}[S_i]$$

which induces

$$\mathbb{Z}[S] \to \varprojlim \mathbb{Z}[S_i] = \mathbb{Z}[S]^{\blacksquare}$$

Def: $S = \varprojlim S_i$

$$\mathbb{Z}[S]^{\blacksquare} := \varprojlim_i \mathbb{Z}[S_i]$$

lim is taken in Cond(Ab).

<u>Recall</u>: $\mathbb{Z}[S]$ is defined by

$$\mathbb{Z}[S] = \left(u \longmapsto \mathbb{Z}[\mathrm{Hom}_{Top}(u,S)]\right)^{sh}.$$

More importantly, $\mathbb{Z}[-] : \mathrm{Cond}(\mathrm{Set}) \to \mathrm{Cond}(\mathrm{Ab})$ is the left adjoint to the forgetful functor.

For each i, we have a projection $S \twoheadrightarrow S_i$, giving a map

$$\mathbb{Z}[S] \longrightarrow \mathbb{Z}[S_i]$$

in the category Cond(Ab). Taking lim yields a map

$$\mathbb{Z}[S] \to \lim_i \mathbb{Z}[S_i] = \mathbb{Z}[S]^{\blacksquare}.$$

Def: • $A \in \mathrm{Cond}(\mathrm{Ab})$ is <u>solid</u> if for all profinite S,

$$\mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare}, A) \longrightarrow \mathrm{Hom}(\mathbb{Z}[S], A) \simeq \mathrm{Hom}(S, A) \simeq A(S)$$

is a bijection.

• $C \in D(\mathrm{Cond}(\mathrm{Ab}))$ is <u>solid</u> if for all profinite S,

$$R\mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare}, C) \longrightarrow R\mathrm{Hom}(\mathbb{Z}[S], C) \simeq R\Gamma(S, C) \quad (*)$$

is an isomorphism.

<u>Rmk</u>: We will see that $(*)$ holds internal RHom as well.

<u>Def</u>: (i) A condensed abelian group A is <u>solid</u> if for all S profinite,

$$\mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare}, A) \to \mathrm{Hom}(\mathbb{Z}[S], A) = \mathrm{Maps}_{\mathrm{Cond}(\mathrm{Set})}(S, A)$$

is an isomorphism. That is, we have the universal property

$$\begin{tikzcd} S \arrow[d] \arrow[dr] & \\ \mathbb{Z}[S]^{\blacksquare} \arrow[r, dashed, "\exists!"'] & A \end{tikzcd}$$

(ii) For $C \in D(\mathrm{Cond}(\mathrm{Ab}))$, we call C <u>solid</u> if for all S profinite,

$$R\mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare}, C) \longrightarrow R\mathrm{Hom}(\mathbb{Z}[S], C) = R\Gamma(S, C).$$

is an isomorphism in $D(\mathrm{Ab})$.

<u>Rmk</u>: It is not yet clear that $A \in \mathrm{Cond}(\mathrm{Ab})$ satisfying (i) implies that $A[0] \in D(\mathrm{Cond}(\mathrm{Ab}))$ satisfies (ii), but we will prove it.

We now compute $\mathbb{Z}[S]^{\blacksquare}$ more explicitly. Write $S = \varprojlim S_i$. We have

$$\mathbb{Z}[S]^{\blacksquare} = \varprojlim \mathbb{Z}[S_i]$$

What is $\mathbb{Z}[S_i]$? We have

$$\mathbb{Z}[S_i] \simeq \mathrm{Hom}(\mathrm{Hom}(\mathbb{Z}[S_i], \mathbb{Z}), \mathbb{Z}) \quad \Big\} \text{ finite rank free abelian group } \mathbb{Z}[S_i] \text{ is isomorphic to its double dual.}$$

$$= \mathrm{Hom}(\underbrace{C(S_i, \mathbb{Z})}_{\text{continuous maps}}, \mathbb{Z})$$

So,

$$\mathbb{Z}[S]^{\blacksquare} \simeq \varprojlim \underline{\mathrm{Hom}}(C(S_i, \mathbb{Z}), \mathbb{Z})$$

$$\simeq \underline{\mathrm{Hom}}(C(S, \mathbb{Z}), \mathbb{Z})$$

Integration: Let $A \in \mathrm{Cond}(\mathrm{Ab})$ and let $f : S \to A$ be a map from a profinite set S. If A is solid, we get an extension

$$\begin{tikzcd} \mathbb{Z}[S]^{\blacksquare} \arrow[dr, dashed, "\exists!"] & \\ S \arrow[u] \arrow[r] & A \end{tikzcd}$$

This is a morphism

$$\underline{\mathrm{Hom}}(C(S,\mathbb{Z}),\mathbb{Z}) \longrightarrow A$$

Evaluate on $*$ to get a map

$$\underbrace{M(S,\mathbb{Z})}_{\substack{\mathbb{Z}\text{-valued} \\ \text{measures}}} = \mathrm{Hom}_{\mathrm{Ab}}(C(S,\mathbb{Z}),\mathbb{Z}) \longrightarrow \underbrace{A(*)}_{\text{underlying abelian group of } A}$$

So, given a measure $\mu \in M(S,\mathbb{Z})$, we get an element

$$\int f \, d\mu \in A(*)$$

called the integral of f wrt μ.

We return to the computation of $\mathbb{Z}[S]^{\blacksquare} \simeq \underline{\mathrm{Hom}}(C(S,\mathbb{Z}),\mathbb{Z})$.
We need the following

Thm (Bergman): Let S be a profinite set. Then $C(S,\mathbb{Z})$ is a free abelian group!

Cor: Let S be a profinite set. Then

$$\begin{aligned}\mathbb{Z}[S]^{\blacksquare} &\simeq \underline{\mathrm{Hom}}(C(S,\mathbb{Z}),\mathbb{Z}) \\ &\simeq \underline{\mathrm{Hom}}\Big(\bigoplus_I \mathbb{Z},\mathbb{Z}\Big) \\ &\simeq \prod_I \mathbb{Z}.\end{aligned}$$

Prop: $\mathbb{Z}[S]^{\blacksquare}$ is solid as both a module and a complex.

Proof: For any profinite T, we must show that

$$R\underline{\mathrm{Hom}}(\mathbb{Z}[T]^{\blacksquare},\mathbb{Z}[S]^{\blacksquare}) \xrightarrow{\sim} R\underline{\mathrm{Hom}}(\mathbb{Z}[T],\mathbb{Z}[S]^{\blacksquare})$$

We may assume that $\mathbb{Z}[S]^{\blacksquare} \simeq \mathbb{Z}$.

RHS: $$\begin{aligned}R\underline{\mathrm{Hom}}(\mathbb{Z}[T],\mathbb{Z}) &\simeq R\Gamma(T,\mathbb{Z}) \quad (H^i(T,\mathbb{Z})=0 \text{ for } i>0) \\ &\simeq H^0(T,\mathbb{Z}) \\ &= C(T,\mathbb{Z}) \\ &= \bigoplus_J \mathbb{Z}\end{aligned}$$

LHS: $R\mathrm{Hom}(\mathbb{Z}[T]^{*}, \mathbb{Z}) \simeq R\mathrm{Hom}(\prod_J \mathbb{Z}, \mathbb{Z})$

To compute this trick, we use our computations from last time.

$$0 \to \prod_J \mathbb{Z} \to \prod_J \mathbb{R} \to \prod_J \mathbb{R}/\mathbb{Z} \to 0$$

Apply $R\mathrm{Hom}(-, \mathbb{Z})$:

$$R\mathrm{Hom}(\prod_J \mathbb{R}/\mathbb{Z}, \mathbb{Z}) \longrightarrow R\mathrm{Hom}(\prod_J \mathbb{R}, \mathbb{Z}) \to R\mathrm{Hom}(\prod_J \mathbb{Z}, \mathbb{Z}) \to$$

This is a distinguished triangle in $D(Ab)$. We have

(1) $R\mathrm{Hom}(\prod_J \mathbb{R}, \mathbb{Z}) \simeq R\mathrm{Hom}_{\mathbb{R}}(\prod_J \mathbb{R}, \underbrace{R\mathrm{Hom}(\mathbb{R}, \mathbb{Z})}_{=0})$

$= 0.$

(2) $R\mathrm{Hom}(\prod_J \mathbb{R}/\mathbb{Z}, \mathbb{Z}) \simeq \bigoplus_J \mathbb{Z}[-1].$

Thus,

$$R\mathrm{Hom}(\prod_J \mathbb{Z}, \mathbb{Z}) \simeq R\mathrm{Hom}(\prod_J \mathbb{R}/\mathbb{Z}, \mathbb{Z})[1]$$

$$\simeq \bigoplus_J \mathbb{Z}[-1][1]$$

$$\simeq \bigoplus_J \mathbb{Z}$$

$\square$

§ 2 The category of solid abelian groups

Our goal now is to show that the category of solid abelian groups has good properties, and that we have a good "solidification" functor.

Thm :

(i) • The full subcategory $\mathrm{Solid} \subseteq \mathrm{Cond}(\mathrm{Ab})$ is abelian, and closed under all limits, colimits, and extension.

• The objects $\mathbb{Z}[S]^{\blacksquare} \simeq \prod_I \mathbb{Z}$ form a family of compact projective generators of solid.

• The inclusion

$$\mathrm{Solid} \hookrightarrow \mathrm{Cond}(\mathrm{Ab})$$

has a left adjoint

$$\mathrm{Cond}(\mathrm{Ab}) \longrightarrow \mathrm{Solid}$$
$$M \longmapsto M^{\blacksquare}$$

It is the unique colimit preserving extension of $(\mathbb{Z}[S] \mapsto \mathbb{Z}[S]^{\blacksquare})$.

(ii) • The derived functor $D(\mathrm{Solid}) \to D(\mathrm{Cond}(\mathrm{Ab}))$ is fully faithful. The image consists of the solid complexes.

• An object $C \in D(\mathrm{Cond}(\mathrm{Ab}))$ is solid iff $H^i(C)$ is solid $\forall i$.

- The inclusion $D(\mathrm{Solid}) \hookrightarrow D(\mathrm{Cond}(\mathrm{Ab}))$ has a left adjoint, given by the left derived functor of $M \mapsto M^{\blacksquare}$.

We need a lemma, which will be proved later

<u>Lemma</u>: Let S be ~~finite~~ extremally disconnected. Suppose that $Y, Z \in \mathrm{Cond}(\mathrm{Ab})$ can expressed as direct sums of objects of the form $\mathbb{Z}[T]^{\blacksquare}$ (T profinite). Let $f: Y \to Z$, $K = \ker f$. Then,

$$R\mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare}, K) \simeq R\mathrm{Hom}(\mathbb{Z}[S], K).$$

(i.e. K is solid).

Assume this lemma. We now prove the theorem.

<u>Proof</u>: Since being solid is a "mapping into" property, Solid is closed under kernels and limits. We must show that cokernels exist. Let

$$f: Y \longrightarrow Z$$

be a map of solid abelian groups.

1. Choose a surjection $\underbrace{\bigoplus_{i \in I} \mathbb{Z}[S_i]}_{\text{these generate } \mathrm{Cond}(\mathrm{Ab})} \twoheadrightarrow Z$.

Replace Y by

$$Y \times_Z \bigoplus_{i \in I} \mathbb{Z}[S_i].$$

This allows us to assume that $Z \simeq \bigoplus_{i \in I} \mathbb{Z}[S_i]$.

② Replace Y by a surjection

$$\bigoplus_{j \in J} \mathbb{Z}[T_j] \twoheadrightarrow Y.$$

Thus we may assume $Y = \bigoplus_{j \in J} \mathbb{Z}[T_j]$

We are now in the situation of the lemma. We have an exact sequence

$$0 \to K \to Y \to Z \to C \to 0 \quad (*)$$

in Cond(Ab). By the lemma,

$$R\mathrm{Hom}(\mathbb{Z}[\underset{\sim}{S}], K) \xleftarrow{\sim} R\mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare}, K)$$

for any S

By solidity of Y, Z, we also have

$$R\mathrm{Hom}(\mathbb{Z}[S], Y) \simeq R\mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare}, Y)$$

$$R\mathrm{Hom}(\mathbb{Z}[S], Z) \simeq R\mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare}, Y)$$

Apply $R\mathrm{Hom}(\mathbb{Z}[S], -)$ and $R\mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare}, -)$ to $(*)$ to yield

$$\begin{tikzcd} R\mathrm{Hom}(\mathbb{Z}[S],K) \arrow[r] \arrow[d, "\wr"] & R\mathrm{Hom}(\mathbb{Z}[S],Y) \arrow[r] \arrow[d, "\wr"] & R\mathrm{Hom}(\mathbb{Z}[S],Z) \arrow[r] \arrow[d, "\wr"] & R\mathrm{Hom}(\mathbb{Z}[S],C) \arrow[d] \\ R\mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare},K) \arrow[r] & R\mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare},Y) \arrow[r] & R\mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare},Z) \arrow[r] & R\mathrm{Hom}(\mathbb{Z}[S],C) \end{tikzcd}$$

By the 5-lemma, C is solid.

In fact we showed that $C[0] \in D(\mathrm{Cond}(Ab))$ is solid.

- Now, given any $Q \in$ Solid, we can write Q as a quotient

$$Y \to Z \to Q \to 0$$

for $Y, Z \in \mathrm{Cond}(Ab)$. By the same trick, we may assume that Y, Z are direct sums of $\mathbb{Z}[S]^{\blacksquare}$'s. Thus,

$$\mathrm{Solid} = \left\{ Q \in \mathrm{Cond}(Ab) \;\middle|\; \begin{array}{l} Q \text{ can be expressed as a cokernel} \\ \bigoplus \mathbb{Z}[S_i]^{\blacksquare} \to \bigoplus \mathbb{Z}[T_j]^{\blacksquare} \to Q \end{array} \right\}$$

This shows that Solid is stable under extensions and direct sums.

Thus, Solid is stable under all colimits.

- Each $\mathbb{Z}[S]^{\blacksquare} \in$ Solid is projective, since

$$\mathrm{Hom}_{\mathrm{Solid}}(\mathbb{Z}[S]^{\blacksquare}, -) \simeq \mathrm{Hom}_{\mathrm{Cond}(Ab)}(\mathbb{Z}[S], -) \text{ is exact}$$

for $S \in \mathrm{ED}$.

Since any $Q \in$ Solid admits a surjection from a direct sum of such objects, they form a family of compact (because $\mathbb{Z}[S]$ is compact) projective generators.

- The existence of the left adjoint

$$\begin{array}{ccc} \mathrm{Cond}(Ab) & \to & \mathrm{Solid} \\ \rotatebox{90}{$\in$} & & \\ A & \longmapsto & A^{■} \end{array}$$

now follows from the adjoint functor theorem

- Now we claim that

$$D(\mathrm{Solid}) \to D(\mathrm{Cond}(Ab))$$

is fully faithful. We must check that

$$R\underset{D(\mathrm{Solid})}{\mathrm{Hom}}(M, N) \longrightarrow R\underset{D(\mathrm{Cond}(Ab))}{\mathrm{Hom}}(M, N)$$

is an isomorphism. We can assume that

$$M = \mathbb{Z}[S]^{■}$$

We can also assume that N is concentrated in one degree.

So, we must show that

$$\mathrm{Ext}^i_{\mathrm{Solid}}(\mathbb{Z}[S]^{■}, N) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathrm{Cond}(Ab)}(\mathbb{Z}[S], N)$$

For $i > 0$, both sides are 0. If $i = 0$, both agree by solidity of N.