1 Recap on cohomology

For S a compact Hausdorff space, we now have two definitions of the cohomology of S with coefficients in an abelian group A.

(1) Consider the constant sheaf $\underline{A}$ on S. We define

$$H^i_{sheaf}(S, A) := H^i(R\Gamma(S, \underline{A})) \qquad \text{(sheaf cohomology)}$$

(2) Consider the constant sheaf $\underline{A}$ on Comp (:= category of compact Hausdorff spaces, with our usual site structure), and view S as an object of Comp. Define

$$H^i_{cond}(S, A) := H^i(R\Gamma(S, \underline{A})) \qquad \text{(condensed cohomology)}$$

First goal for today: For any $S \in$ Comp,

$$\boxed{H^i_{cond}(S, A) \cong H^i_{sheaf}(S, A)}$$

2. Topos theoretic preliminaries

Def^n: A *topos* is a category of the form

$$\mathcal{T} = Sh(\mathcal{C})$$

for $\mathcal{C}$ a (small) site. A geometric morphism of topoi

$$\mathcal{T}_1 \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathcal{T}_2$$

is a pair of functors f^*, f_* such that

(i) f^* is left adjoint to f_*.

(ii) f^* preserves finite limits (i.e. f^* is exact)

<u>Construction</u>: Given a subcanonical site $\mathcal{C}$ and $X \in \mathcal{C}$, we may form the <u>overcategory</u> $\mathcal{C}_{/X}$:

$$Ob(\mathcal{C}_{/X}) = \{ Y \to X \in Mor(\mathcal{C}) \}$$

$$Hom(Y_1, Y_2) = \left\{ \begin{tikzcd} Y_1 \arrow[rr] \arrow[dr] & \arrow[d, phantom, "\circlearrowleft"] & Y_2 \arrow[dl] \\ & X & \end{tikzcd} \in Hom_{\mathcal{C}}(Y_1, Y_2) \right\}$$

It is naturally a site. So, we may form the topos

$$Sh(\mathcal{C}_{/X})$$

On the other hand, we can view X as an object of $\mathcal{T} = Sh(\mathcal{C})$. There is an equivalence:

$$\mathcal{T}_{/X} \simeq Sh(\mathcal{C}_{/X}).$$

In particular, $\mathcal{T}_{/X}$ is once again a topos.

<u>Generalization</u>: We can just let $F \in Sh(\mathcal{C})$ be general (i.e. not necessarily in the image of the Yoneda embedding $\mathcal{C} \hookrightarrow Sh(\mathcal{C})$). Then $\mathcal{C}_{/F}$ still makes sense:

$$Ob(\mathcal{C}_{/F}) = \{ Y \xrightarrow{f} F \mid Y \in \mathcal{C}, f \in Hom_{Sh(\mathcal{C})}(Y, F) \simeq F(Y) \}$$

and it is naturally a site.

<u>Thm (Fundamental theorem of topos theory)</u>:

$$\mathcal{T}_{/F} \simeq Sh(\mathcal{C}_{/F})$$

In particular, $\mathcal{T}_{/F}$ is a topos. It is the <u>overtopos</u> of F.

<u>Remarks</u> (1) There is a natural geometric morphism

$$\mathcal{T}_{/F} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathcal{T}$$

$$f_*\left(\begin{matrix} G \\ \downarrow \\ F \end{matrix}\right) = G$$

$$f^*(G) = \begin{matrix} G \times_1 F \\ \downarrow \\ F \end{matrix}$$

(2) Every topos has a terminal object 1. Then the geometric morphism

$$\mathcal{T}_{/1} \rightleftarrows \mathcal{T}$$

is an equivalence of topoi

3. <u>Cohomology</u> <u>in a topos</u>

If $\mathcal{T} = Sh(\mathcal{C})$, then

$$\underbrace{\mathcal{T}^{ab}}_{\substack{\text{abelian group} \\ \text{objects in } \mathcal{T}}} \simeq \underbrace{Ab(\mathcal{C})}_{\substack{\text{abelian sheaves} \\ \text{on } \mathcal{C}.}}$$

We may also see the global sections functor internally:

$$\Gamma := Hom_{\mathcal{T}}(\underbrace{1}_{\text{terminal}}, -).$$

Example If X is a space, then

$$\Gamma : Sh(X) \longrightarrow Set$$

is given by

$$\Gamma(\mathcal{F}) = Hom_{Sh(X)}(*, \mathcal{F}) = \mathcal{F}(X),$$

Now we may define cohomology in $\mathcal{T}$.

Def: For $F \in \mathcal{T}$,

$$H^i(F) := H^i R\Gamma(F).$$

Rmk This doesn't depend on the site $\mathcal{C}$ we use to write $\mathcal{T} = Sh(\mathcal{C})$. Motto: one topos, one cohomology functor.

Def: The point is the topos $Sh(*) = Set$. There is a natural geometric morphism

$$\begin{tikzcd} \mathcal{T} \arrow[r, shift right, "p_*"'] & * \arrow[l, shift right, "p^*"'] \end{tikzcd}$$

$$p_*(F) = \Gamma(F) = Hom(1, F).$$

$$\underset{\substack{\text{in} \\ Set}}{p^*(S)} = \underline{S} \text{ (constant sheaf associated to } S).$$

Thm. Let

$$\begin{tikzcd} \mathcal{T}_1 \arrow[r, shift right, "f_*"'] & \mathcal{T}_2 \arrow[l, shift right, "f^*"'] \end{tikzcd}$$

be a geometric morphism, with f^* fully faithful. For any $F \in \mathcal{T}_2^{ab}$, $\exists$ a canonical isomorphism

$$H^i(f^*F) \simeq H^i(F) \qquad \forall i$$

Proof: We have a diagram

$$\begin{tikzcd} \mathcal{T}_1^{ab} \arrow[rr, shift right, "f_*"'] \arrow[dr, bend right, "p_{1*}"'] & & \mathcal{T}_2^{ab} \arrow[ll, shift right, "f^*"'] \arrow[dl, bend left, "p_{2*}"] \\ & *^{ab} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

f^* preserves finite limits and colimits, so it is exact. Since f^* is fully faithful, the counit

$$id_{\mathcal{T}_2^{ab}} \to f_* f^*$$

is an isomorphism. By exactness of f^*, we have an isomorphism

$$id \simeq Rf_* f^*.$$

Thus,

$$Rp_{2*}(F) \simeq Rp_{2*} Rf_* f^* F \simeq Rp_{1*} f^* F.$$

Take H^i on both sides. $\square$

4. Back to condensed sets

Let us now properly interpret H^i_{sheaf} and H^i_{cond}. Let $S \in Comp$.

(1) Consider the overtopos

$$Sh(Comp/S) \simeq Sh(Comp)/_S = Cond(Set)/_S.$$

The global sections functor is

$$\Gamma : Cond(Set)/_S \longrightarrow Set$$
$$F \longrightarrow S \longmapsto F(S)$$

So,

$$H^i_{cond}(S, A) \cong H^i_{Sh(Comp/S)}(\underline{A}) \qquad \underline{A} = p^*A.$$

(2) $H^i_{sheaf}(S, A) = H^i_{Sh(S)}(\underline{A}) \qquad \underline{A} = p^*A.$

If we produce a geometric morphism

$$\begin{tikzcd} Sh(S) \arrow[r, shift right, "f_*"'] & Sh(Comp/S) \arrow[l, shift right, "f^*"'] \end{tikzcd}$$

Then we will automatically have $f^*\underline{A} = \underline{A}$, so if f^* is fully faithful,

$$H^i_{sheaf}(S, A) \cong H^i_{cond}(S, A).$$

More generally, suppose that S is LCH. View it as a condensed set. We can form the topos $Sh(Comp/S) \simeq Sh(Comp)/\underline{S}$
$= Cond(Set)/\underline{S}$
and define $H^i_{cond}(S, A)$. We will show

$$H^i_{cond}(S, A) \simeq H^i_{sheaf}(S, A)$$

by exhibiting a shape equivalence

$$Sh(S) \underset{f_*}{\overset{f^*}{\rightleftarrows}} Sh(Comp/S)$$

We will do it by introducing a third topos $Sh(Comp(S))$.

<u>Step 1</u>: Lurie's description of $Sh(S)$.

Let S be LCH. Consider the poset/category

$Comp(S)$:= poset of compact subsets of S under inclusion.

Note that $Comp(S)$ has a natural site structure: a covering is a finite family

$$\{S_i \hookrightarrow S\}_{i=1}^{n}$$

with $S = \bigcup_{i=1}^{n} S_i$.

Thm (Lurie, HTT 7.3.4) Let S be LCH. Then

$$\begin{aligned} e^* : Sh(S) &\longrightarrow Fun(Comp(S)^{op}, Set) \\ F &\longmapsto [K \mapsto \operatorname*{colim}_{U \supset K} F(U)] \end{aligned}$$

has the following properties.

(1) e^* is fully faithful.

(2) e^* factors through $Sh(Comp(S))$

(3) The essential image of e^* consists of <u>overconvergent</u> sheaves.

<u>Def</u>. $F \in Sh(Comp(S))$ is <u>overconvergent</u> if $\forall K \in Comp(S)$,

$$\operatorname*{colim}_{K' \supset\supset K} F(K') \to F(K)$$

is an iso.

(4). The functor $e^*(Sh(S)) \hookrightarrow Sh(Comp(S))$ has a right adjoint

$$\begin{aligned} e_* : Sh(Comp(S)) &\to Sh(S) \\ G &\mapsto e_* G \end{aligned}$$

$$e_* G(K) = \operatorname*{colim}_{K' \supset\supset K} G(K').$$

(5) e^* is left exact (filtered colimits are exact in set)

So, we have

$$\begin{tikzcd} Sh(Comp(S)) \arrow[r, shift right, "e_*"'] \arrow[r, phantom, "\perp"] & Sh(L) \arrow[l, shift right, "e^*"'] \end{tikzcd}$$

Step 2 Now we must relate Sh(Comp(S)) with

$$\mathrm{Cond}(\mathrm{Set})_{/S} \simeq \mathrm{Sh}(\mathrm{Comp}_{/S})$$

There is an obvious functor

$$\begin{aligned} i : \mathrm{Comp}(S) &\hookrightarrow \mathrm{Comp}_{/S} \\ K \subseteq S &\longmapsto K \to S \end{aligned}$$

It yields a geometric morphism

$$\begin{tikzcd} \mathrm{Sh}(\mathrm{Comp}(S)) \arrow[r, shift right, "i_*"'] \arrow[r, phantom, "\perp"] & \mathrm{Sh}(\mathrm{Comp}_{/S}) \arrow[l, shift right, "i^*"'] \end{tikzcd}$$

All that remains is to see that i^* is fully faithful.
Since i satisfies the covering lifting property, right Kan extension along

$$i^{op} : \mathrm{Comp}(S)^{op} \hookrightarrow (\mathrm{Comp}_{/S})^{op}$$

preserves sheaves, giving a functor

$$i^! : \mathrm{Sh}(\mathrm{Comp}(S)) \to \mathrm{Sh}(\mathrm{Comp}_{/S}).$$

It is fully faithful because i^{op} is, and right adjoint to i_*. Hence

$$i^! \text{ fully faithful} \implies i^* \text{ fully faithful}. \qquad \square$$