Globalization of Solid Modules & Coherent Cohomology w/ Compact Support

§1: Relativizing Theorems (1)-(3) from last week.

Def^n

Let $R \to A$ be a map of finitely generated $\mathbb{Z}$-algebras. We define $(A,R)_\square$ to be the pre analytic ring whose underlying condensed ring is A (as a discrete ring), &

$$(A,R)_\square[S] := \underbrace{R_\square[S]}_{= \varprojlim R[S_i]} \otimes_R A$$

Remark

$$R_\square[S] \overset{L}{\otimes}_R A \simeq R_\square[S] \otimes_R A \quad \text{see by viewing}$$

$$R_\square[S] \simeq \prod_I R$$

Theorem 0

$(A,R)_\square$ is analytic.

Rephrase the theorems from last time in the relative setting.

Note

The initial reduction ~~from~~ to the case $A = \mathbb{Z}[t]$ from last time can be done in the relative setting, to $A = R[t]$, & then similar proofs go through.

Theorem R1

Let $R \to S \to A$ be maps of finitely generated $\mathbb{Z}$-algs.

(R1.1) $j_* : D((A,S)_\square) \hookrightarrow D((A,R)_\square)$ forgetful admits a left adjoint

$$j^* := - \overset{L}{\otimes}_{(A,R)_\square} (A,S)_\square$$ which in turn admits a fully faithful left adjoint $j_!$

$$\begin{tikzcd} D((A,R)_\square) \arrow[rr, "j^*" description] & & D((A,S)_\square) \arrow[ll, bend right, "j_!"', "\perp"] \arrow[ll, bend left, "j_*", "\perp"'] \end{tikzcd}$$

(R1.2) <u>Projection</u> <u>Formula</u>: $M \in D((A,R)_\square)$-module

$$j_! j^*(M) \simeq M \overset{L}{\otimes}_{(A,R)_\square} j_! A$$

this is enough!

<u>Def^n</u> (Assuming Theorem R1) can define relative compactly supported cohomology:

$$f : \operatorname{Spec} A \longrightarrow \operatorname{Spec} R \quad \text{f.t.} / \mathbb{Z}$$

$$\begin{tikzcd} D(A_\square) = D((A,A)_\square) \arrow[r, "j_!"] \arrow[rr, bend right, "f_!"'] & D((A,R)_\square) \arrow[r, "\text{Forget}"] & D(R_\square) \end{tikzcd}$$

Theorem R2

(R2.3) $f_!$ commutes w/ direct sums & satisfies a projection formula:

$M \in D(R_\square)$, $N \in D(A_\square)$

$$M \otimes^L_{R_\square} f_! N \simeq f_!\left((M \otimes_{R_\square} A_\square) \otimes^L_{A_\square} N \right).$$

If f has finite Tor-dimension, then

(R2.1) $f_!$ admits a right adjoint

$$\cancel{f^! = R\mathrm{Hom}((A,R)_\square, -) \otimes^L_{(A,R)_\square} A_\square}$$

(R2.2) $f_!$ preserves compact objects

Theorem R3

(If f has finite Tor-dimension:)

(R3.1) $f^!(R) \in D(A_\square)$ is discrete & bounded to the left complex of f.gen'd A-modules. If f has finite Tor dimension ~~[illegible]~~ $f^!R$ is bounded.

(R3.2) If f has finite Tor dimension, $f^!$ is given by

$$f^!(M) = (M \otimes^L_{R_\square} A_\square) \otimes^L_{A_\square} f^! R.$$

(R3.3) If f has finite Tor dimension $f^!$ preserves discrete objects.

(R3.4) If f is a complete intersection, $f^!(R)$ is invertible.

Theorem (R4)

$$\operatorname{Spec} A \xrightarrow{f} \operatorname{Spec} S \xrightarrow{g} \operatorname{Spec} R \quad \text{s.t. } /\mathbb{Z}.$$

$$(g \circ f)_! \simeq g_! \circ f_!$$

$$(g \circ f)^! \simeq f^! \circ g^!$$

A word on R4

$$\begin{tikzcd}
D(A_\square) \arrow[r, hook] \arrow[dr, "f_!"'] \arrow[ddrr, dashed, bend left=60, "(g\circ f)_!"] & D((A,S)_\square) \arrow[r, hook, "j_!"] \arrow[d, "\text{forget}"] \arrow[dr, phantom, "①"] & D((A,R)_\square) \arrow[d, "\text{forget}"] \\
 & D(S_\square) \arrow[r, "\hat{j}_!"] \arrow[dr, "g_!"'] & D((S,R)_\square) \arrow[d, "\text{forget}"] \\
 & & D(R_\square)
\end{tikzcd}$$

Suffices to show ① commutes

Certainly true replacing $j_!$ w/ j_*. Is that enough since adjointing is formal?

§2: Globalization.

Want: A geometric framework so that $f: X \longrightarrow Y$ get

some $\underbrace{f_! : D(X_\square)}_{\uparrow \text{ what is this?}} \longrightarrow D(Y_\square)$

Know how to do this when X corresponds to (pairs of) f. gen'd $\mathbb{Z}$-algebras.

Lemma $R \xrightarrow{\phi} A$ map of f. gen'd $\mathbb{Z}$-algs.

$A^+ = \widetilde{\phi(R)}$ = Integral Closure of $\phi(R)$ in A.

Then the analytic rings

$$(A, R) \xleftarrow{\sim} (A, A^+)$$

Pf/ $R \xrightarrow{\text{finite}} A^+ \longrightarrow A$ $\quad$ A^+ a finite R-module

$$(A, R)_\square [S] = R_\square[S] \otimes_R A$$

$$= \underbrace{R_\square[S] \otimes_R A^+}_{=(\prod_I R) \otimes_R A^+ \;\cong\; \prod_I A^+ \text{ (finite)}} \otimes_{A^+} A \simeq A^+[S] \otimes_{A^+} A \simeq (A, A^+)_\square [S]$$

$\not\!\!\!\emptyset$

So our functor $(R \to A) \longmapsto \{D(A,R)_0\}$ is most naturally sourced in the category of pairs (A, A^+) w/ A f.g. $/\mathbb{Z}$ & A^+ a f.g. int. closed subalg. $\underset{A}{\overset{\cap\!|}{A^+}}$

This naturally puts us in the setting of Huber's Theory.

Def^n

A discrete Huber Pair is a pair (A, A^+) of discrete rings w/ $A^+ \subseteq A$ integrally closed.

Get a functor

$$\begin{pmatrix}\text{Discrete Huber} \\ \text{Pairs}\end{pmatrix} \longrightarrow \begin{pmatrix}\text{Analytic} \\ \text{Rings}\end{pmatrix}$$

$$(A, A^+) \longmapsto \underbrace{(A, A^+)_\square}_{\uparrow \text{ not f. gen'd...}} := \varinjlim_{\substack{(B,B^+) \to (A,A^+) \\ \uparrow \\ \text{f. gen'd DHP}}} (B, B^+)_\square$$

Remark:

Huber pairs exist more generally for topological rings.

A = topological ring w/ $A_0 \subseteq A$ open & I-adic for $I \subseteq A_0$.

A^+ = open integrally closed $\underset{\text{bounded.}}{\wedge}$ subring

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Naturally includes adic rings & their generic fibers.

Def^n Given a Discrete huber pair (A, A^+) one can form

$$\mathrm{Spa}(A, A^+) := \{ |\cdot| : A \longrightarrow \Gamma \cup \{0\} \mid |A^+| \leq 1 \} / \cong$$

(w/ Γ a totally ordered abelian group (of elts > 0)
$|\cdot|$ nonarchimedean absolute value
$|0| = 0$, $|1| = 1$, $|xy| = |x| \cdot |y|$, $|x+y| \leq \max\{|x|, |y|\}$.)

Notation Convention

$x \in \mathrm{Spa}(A, A^+)$ is a valuation on A.

We write instead of $x(f) =: |f(x)|$ ← This is often how we get points of Spa

Remarks

① More general Huber rings we require $|\cdot|$ to be cts

② Why should valuations be points?

ⓐ Peter's universal compactification argument.

ⓑ X = curve. $x \in X$ a point, get a valuation on

$$\begin{array}{ccccc} K(X) & \longrightarrow & \mathbb{Z} & \xrightarrow{e^{-x}} & \mathbb{R}_{>0} \\ f & \longmapsto & \mathrm{ord}_x(f) & & \uparrow \\ \cup\!| & & & & \\ A(X) & \longrightarrow & & & [0,1] \end{array}$$

Recover $A(X)$ (thus X) from the correct collection of vals on $K(X)$

(c) Points of $\overline{\mathbb{D}}^1 = \mathrm{Spa}(K\langle X\rangle, \mathcal{O}_K\langle X\rangle)$ K nonarch field.

Should be an analytic space whose ~~points~~ ~~are~~ functions are convergent power series.

E.g. $x \in \mathcal{O}_K$. $f \in K\langle X\rangle$ can be evaluated @ x.

& $f(x) \in K$ & $|f(x)| \in \mathbb{R}$ And indeed

$$* \quad f \longmapsto |f(x)| \text{ is a valuation of } K\langle X\rangle$$

which is ≤ 1 on $\mathcal{O}_K\langle X\rangle \implies$ a $x \in \overline{\mathbb{D}}^1$.

(d) K-points of $\overline{\mathbb{D}}^1$.

$* = \mathrm{Spa}(K, \mathcal{O}_K)$

<u>$(K, \mathcal{O}_K)$-points</u>:

$$* \longrightarrow \overline{\mathbb{D}}^1$$

$$\Updownarrow$$

$$(K\langle X\rangle, \mathcal{O}_K\langle X\rangle) \longrightarrow (K, \mathcal{O}_K)$$

$$\Updownarrow$$

$$\begin{tikzcd} K\langle X\rangle \arrow[r, "\phi"] & K \\ \mathcal{O}_K\langle X\rangle \arrow[u, phantom, "\cup"] \arrow[r, "\phi"] & \mathcal{O}_K \arrow[u] \end{tikzcd}$$

determined by $\phi(X) \in \mathcal{O}_K$ (by continuity).

so $\overline{\mathbb{D}}^1(K, \mathcal{O}_K) = \mathcal{O}_K$.

Topologizing $Spa(A, A^+) = X$

Open Sets Can Be Defined By Inequalities

Basis given by

$$U\left(\frac{g_1, \cdots, g_n}{f}\right) = \{x \in X \mid |g_i(x)| \leq |f(x)| \neq 0\}$$

as $g_1, \cdots, g_n, f \in A$ vary.

This makes X a spectral space ($\simeq$ Spec some ring)

Notice

$$Spa(A, A^+) \subseteq Spa(A, \tilde{\mathbb{Z}}) =: Spv(A)$$

open.

All valuations since $|\tilde{\mathbb{Z}}| \leq 1$ by sth ultralytic.

Prop

$$\left\{\begin{array}{c}\text{Integrally} \\ \text{closed} \\ A^+ \subseteq A\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{Opens } U \subseteq Spa(A, \tilde{\mathbb{Z}}) \\ \text{s.t.} \\ U = \bigcap_{f \in S \subseteq A} U\left(\frac{f}{1}\right)\end{array}\right\}$$

$$A \longmapsto U = Spa(A, A^+) = \bigcap_{f \in A^+} U\left(\frac{f}{1}\right)$$

$$A^+ := \left\{\begin{array}{l} f \in A : \\ \forall x \in U \\ |f(x)| \leq 1 \end{array}\right\} \longleftarrow\!\!\shortmid U$$

Proof

1) $A^+(U)$ is integrally closed. Indeed, if $f \in A$ int $/A^+(U)$
$\Rightarrow \|f(x)\| \le 1$ $\forall x \in U$ by ultrametric $\Rightarrow f \in A^+(U)$

2) Take $U = \bigcap_{f \in I} U(\tfrac{f}{1})$ on RHS. $A^+ = \overline{\langle f \mid f \in I \rangle} \subseteq A$

$\Rightarrow \mathrm{Spa}(A, A^+) = U$

3) $\underset{\substack{\Psi \\ f}}{A^+(\mathrm{Spa}(A,A^+))} = \{ f \in A \mid \forall x \in \mathrm{Spa}(A,A^+),\ |f(x)| \le 1 \} \supseteq A^+$

$f \notin A^+ \Rightarrow f \notin A^+[1/f] \subseteq A[1/f]$

$$\begin{array}{l} \cup\! \mathsf{I} \\ \frac{1}{f} \in P \leftarrow \text{prime} \\ \cup\! \mathsf{I} \\ q \leftarrow \text{minimal prime.} \end{array}$$

$$\begin{tikzcd}
\mathrm{Spec}\, K \arrow[r, dashed] \arrow[d] & \mathrm{Spec}\, A[1/f] \arrow[d] \\
\mathrm{Spec}\, V \arrow[r] \arrow[ur] & \mathrm{Spec}\, A^+[1/f]
\end{tikzcd}$$

$$\begin{array}{ccc} \Psi & & \\ x & \longmapsto & p \\ \eta & \longmapsto & q \end{array}$$

Get a valuation on A

$$\begin{tikzcd}
 & A[1/f] \arrow[r] & K \arrow[r, "v"] & {} \\
A^+ \arrow[r] & A^+[1/f] \arrow[r] \arrow[u, phantom, "\cup\mathsf{I}"] & V \arrow[r, "v"] \arrow[u, phantom, "\cup\mathsf{I}"] & {} \\
 & \frac{1}{f} \arrow[r, mapsto] & m \arrow[r, mapsto] & <1
\end{tikzcd}$$

$\Rightarrow |f| > 1$ $\lightning$.

(cont.)

Equivalence of Categories

~~Affinoid~~

$$\begin{pmatrix}\text{Disc Huber}\\ \text{Pairs}\end{pmatrix} \xrightarrow{\text{Spa}} \begin{pmatrix}\text{Affinoid Discrete}\\ \text{adic spaces}\end{pmatrix}$$

$$(A, A^+) \longmapsto \text{Spa}(A, A^+)$$

$$(A, A^+) \to (B, B^+) \longmapsto \begin{array}{c}\text{Spa}(A, A^+)\\ \uparrow\\ \text{Spa}(B, B^+)\end{array}$$

$$\begin{array}{ccccc} A^+ & \to & B^+ & \to & \Gamma_{\leq 1}\\ \cap & & \cap & & \cap\\ A & \to & B & \to & \Gamma \end{array}$$

$$\uparrow$$

$$\text{Spa}(B, B^+):\ \ \begin{array}{ccc} v: B & \to & \Gamma\\ \cup & & \cup\\ B^+ & \to & \Gamma_{\leq 1}\end{array}$$

Recall

For Schemes:

$$\begin{array}{ccc} X & = & \text{Spec}\, A\\ \cup & & \cup\\ D(f) = U & = & \text{Spec}(A_f)\end{array}$$

What does (topological) localization for adic spaces look like algebraically?

Prop: $X = \mathrm{Spa}(A, A^+)$

$$U = U\left(\frac{g_1, \dots, g_n}{f}\right) = \{x \in X \mid |g_i(x)| \le |f(x)| \neq 0\}$$

Then $U \simeq \mathrm{Spa}\left(A[1/f], \overline{A^+\left[\frac{g_1}{f}, \dots, \frac{g_n}{f}\right]}\right)$

Specifically

Suppose $(A, A^+) \longrightarrow (B, B')$ induces

$$\begin{tikzcd} \mathrm{Spa}(B,B') = Y \arrow[r, "\varphi"] \arrow[dr] & X \\ & U \arrow[u, hook] \end{tikzcd}$$

Then

$$\begin{tikzcd} (A, A^+) \arrow[r, "\varphi^\#"] \arrow[d] & (B, B^+) \\ \left(A\left[\frac{1}{f}\right], \overline{A\left[\frac{g_1}{f}, \dots, \frac{g_n}{f}\right]}\right) \arrow[ur, dotted, "\exists! \alpha"'] & \end{tikzcd}$$

& $\mathrm{Spa}\left(\left(A[1/f], \overline{A\left[\frac{g_1}{f}, \dots, \frac{g_n}{f}\right]}\right)\right) \longrightarrow X$ is a homeomorphism onto U.

Proof

First notice $\varphi^\#(f) \in B^*$.

Else, $\exists \quad \varphi^\#(f) \in p \in \operatorname{Spec} B$

&

$$\begin{tikzcd} B \arrow[r] \arrow[rr, bend left, "v_p"] & \operatorname{Frac}(B/p) \arrow[r, "\text{triv}"] & \{0,1\} \end{tikzcd}$$

$$b \longmapsto \begin{cases} 0 & b \in p \\ 1 & \text{else} \end{cases}$$

& $\quad v_p(\varphi^\#(f)) = 0$

$$\begin{array}{c} \| \end{array}$$

$$|\varphi^\#(f)(p)| = |f(\underbrace{\varphi(p)}_{\in U})| \neq 0 \quad \lightning$$

But $\quad \forall \; y \in Y$

$$|\varphi^\#(f)(y)| = |f(\varphi(y))| \neq 0$$

So

$$\begin{tikzcd} A \arrow[rr, "\varphi^\#"] \arrow[dr] & & B \\ & A[1/f] \arrow[ur, "\varphi^\#"'] & \end{tikzcd}$$

Now $\quad \forall \; y \in Y$

$$|\varphi^\#(\tfrac{g_i}{f})(y)| = |(\tfrac{g_i}{f})(\underbrace{\varphi(y)}_{\in U})| \leq 1$$

$$\Longrightarrow \varphi^\#\left(\overline{A^+[\tfrac{g_1}{f}, \ldots, \tfrac{g_n}{f}]}\right) \subseteq B^+$$

In particular, we see that the open set $U = U\left(\frac{g_1, \dots, g_n}{f}\right)$ uniquely determines the discrete Huber pair $(A[1/f], \widetilde{A^+[g_1/f, \dots, g_n/f]})$.

<u>Defn</u> $X = \mathrm{Spa}(A, A^+)$.

Define sheaves $\mathcal{O}_X, \mathcal{O}_X^+$ on X by defining them on a basis opens of the form $U = U\left(\frac{g_1, \dots, g_n}{f}\right)$

$$\mathcal{O}_X(U) = A[1/f]$$
$$\mathcal{O}_X^+(U) = \widetilde{A^+[g_1/f, \dots, g_n/f]}.$$

<u>Propn</u>

1) $\mathcal{O}_X$ & $\mathcal{O}_X^+$ are sheaves.

2) $x \in X$, extends uniquely to a valuation on $\mathcal{O}_{X,x}$.

3) $\mathcal{O}_X^+(U) = \{ f \in \mathcal{O}_X(U) \mid |f(x)| \leq 1 \ \forall x \in U \}$

makes sense on stalks

<u>proof</u>

Let $\varphi : \mathrm{Spec}\, A \longrightarrow \mathrm{Spa}(A, A^+)$

$$p \longmapsto v_p : A \to \mathrm{Frac}(A/p) \xrightarrow{\text{triv}} \{0,1\}$$

This is continuous, & $\varphi^{-1}(\underbrace{U\left(\tfrac{g_1, \dots, g_n}{f}\right)}_{U}) = D(f)$

So $\mathcal{O}_X(U) = A[1/f] = \mathcal{O}_{\mathrm{Spec}\, A}(D(f)) = \mathcal{O}_{\mathrm{Spec}\, A}(\varphi^{-1}(U))$

$\Longrightarrow \mathcal{O}_X = \varphi_* \mathcal{O}_{\mathrm{Spec}\, A}$ & thus a sheaf!

2)

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A[1/f] \arrow[r] \arrow[dl, dotted, "\exists!"] & \operatorname{colim} \mathcal{O}_X(U) = \mathcal{O}_{X,x} \arrow[dll, dotted, "\exists!"] \\ \Gamma_x & & \end{tikzcd}$$

gives (2)

3) True on $(A[1/f], A^+[g_1/f, \dots, g_n/f])$ (already saw).

Back to

1) $\mathcal{O}_X^+ \subseteq \mathcal{O}_X$ now given by a stalkwise condition.

So it's a sheaf!

(glue uniquely in $\mathcal{O}_X$ then check on stalks). $\boxtimes$

<u>Def^n</u>

A discrete adic space is a triple $(X, \mathcal{O}_X, \underset{(v_x)}{(|\cdot(x)|)_{x\in X}})$

where X a top space, $\mathcal{O}_X$ a sheaf of rings, $|\cdot(x)|$ a valuation equiv. class on the stalk $\mathcal{O}_{X,x}$, which is locally

$$\left(\operatorname{Spa}(A,A^+), \mathcal{O}_{\operatorname{Spa}(A,A^+)}, \underset{v_x}{(|\cdot(x)|)_{x\in \operatorname{Spa}(A,A^+)}}\right)$$

for (A,A^+) a discrete Huber pair.

<u>Rmk</u> * A ~~morphism of~~ (pre)-adic space is $(X, \mathcal{O}_X, (v_x)_{x\in X})$

s.t. $\mathcal{O}_{X,x}$ is local w/ max'l ideal $\operatorname{supp}(v_x)$.

* A morphism $f : (X, \mathcal{O}_X, (v_x)_{x\in X}) \longrightarrow (Y, \mathcal{O}_Y, (v_y)_{y\in Y})$ is ...

(of pre-adic spaces

$$\left.\begin{array}{l} f : X \longrightarrow Y \\ f^\# : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X \end{array}\right\} \text{ map of LRS.}$$

So that $f(x) = y$

$$\begin{tikzcd}
\mathcal{O}_{Y,y} \arrow[r] \arrow[d, "v_y"] & \mathcal{O}_{X,x} \arrow[d, "v_x"] \\
\Gamma_y \arrow[r, phantom, "\subseteq"] & \Gamma_x
\end{tikzcd}$$

Given $(X, \mathcal{O}_X, (v_x = |\cdot(x)|))$

$$\mathcal{O}_X^+ : U \longmapsto \{ f \in \mathcal{O}_X(U) \mid |f(x)| \le 1 \ \forall x \in U \}.$$

Then $\mathcal{O}_{X,x}^+ = \{ f \in \mathcal{O}_{X,x} \mid |f(x)| \le 1 \}$ is a valuation ring w/ max'l ideal $\mathfrak{m}_{X,x} = \{ f \in \mathcal{O}_{X,x} \mid |f(x)| < 1 \}$

__Prop__ $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ map of locally ringed spaces is a map of (pre)-adic spaces iff

1) $$\begin{tikzcd}
\mathcal{O}_Y \arrow[r] & f_*\mathcal{O}_X \\
\mathcal{O}_Y^+ \arrow[u] \arrow[r, dashed] & f_*\mathcal{O}_X^+ \arrow[u]
\end{tikzcd}$$

&

2) $\mathcal{O}_Y^+ \longrightarrow f_*\mathcal{O}_Y^+$ is a local map.

In particular, if $X = \mathrm{Spa}(A, A^+)$ & $Y = \mathrm{Spa}(B, B^+)$

$$\mathrm{Hom}(X, Y) = \mathrm{Hom}((B, B^+), (A, A^+))$$

More generally, X affinoid $\mathrm{Hom}(X, Y) = \mathrm{Hom}((B, B^+), (\mathcal{O}_X(X), \mathcal{O}_X^+(X)))$

Proof

$$\begin{tikzcd}
 & & \mathfrak{m}_y \arrow[rrr, "②"] \arrow[dd] & & & \mathfrak{m}_x \arrow[dd] \\
 & \mathcal{O}^+_{Y,y} \arrow[rrr] \arrow[ddl] \arrow[dr] & & & \mathcal{O}^+_{X,x} \arrow[ddl] \arrow[dr] & \\
 & & k(y)^+ \arrow[rrr, "①"] \arrow[ddl] & & & k(x)^+ \arrow[ddl] \\
\mathcal{O}_{Y,y} \arrow[rrr] \arrow[dr] \arrow[dd] & & & \mathcal{O}_{X,x} \arrow[dr] \arrow[dd] & & \\
 & k(y) \arrow[rrr] \arrow[dl, "v_y"] \arrow[drr, "v_x"] & & & k(x) \arrow[dl] & \\
\Gamma_y & & & \Gamma_x & &
\end{tikzcd}$$

Goal valuation rings for $\overset{k(y)^+}{v_y}$ & $\overset{A}{v_x}$ in $k(y)$ agree.

① ~~If~~ $|f(y)| \le |\overset{1}{g(y)}| \Rightarrow |f(x)| \le |\overset{1}{g(x)}| \Rightarrow k(y)^+ \subseteq A$

② $|f(y)| < |\underset{1}{g(y)}| \Rightarrow |f(x)| < |\underset{1}{g(x)}|$

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② ~~Since~~ $f \notin k(y)^+ \Rightarrow |f(y)| > 1$

$\Rightarrow |f(x)| > 1$

$\Rightarrow f \notin A$

So $A \subseteq k(y)^+$

Points on adic spaces

Given $x \in X = \mathrm{Spa}(A, A^+)$

Get

$$\begin{tikzcd}
A^+ \arrow[r] & \mathcal{O}^+_{X,x} \arrow[r] & k(x)^+ \\
A \arrow[u, phantom, "\cap" rotate=90] \arrow[r] \arrow[dr, "x"'] & \mathcal{O}_{X,x} \arrow[u, phantom, "\cap" rotate=90] \arrow[r] \arrow[d, "x"] & k(x) \arrow[u, phantom, "\cap" rotate=90] \arrow[dl, "x"] \\
& \Gamma_x &
\end{tikzcd}$$

So get $(A, A^+) \longrightarrow (k(x), k(x)^+)$

i.e. $\mathrm{Spa}(k(x), k(x)^+) \longrightarrow X$

Conversely Given $\mathrm{Spa}(k, k^+) \longrightarrow X$

may have multiple pts

$\wr$

$$\begin{tikzcd}
A^+ \arrow[r] & k^+ \arrow[dr] & \\
A \arrow[u, phantom, "\cap" rotate=90] \arrow[r] \arrow[rr, dashed, bend right, "2^{?}_x"'] & k \arrow[u, phantom, "\cap" rotate=90] \arrow[r, "?"'] & \Gamma
\end{tikzcd}$$

Choosing a VR = choosing a Valuation

Get $x \in X$.

Slogan

Points $=$ (maps from $\mathrm{Spa}(k, k^+)$ where k^+ is a valuation ring)

2 Special Cases

① $X = \mathrm{Spa}(A, A)$

pts $\leftrightsquigarrow$
$$\begin{tikzcd} K & A \arrow[l] \\ K^+ \arrow[u, hook] & A \arrow[l] \arrow[u, equal] \end{tikzcd} \leftrightsquigarrow \operatorname{Spec} K^+ \longrightarrow \operatorname{Spec} A$$

② $X = \mathrm{Spa}(A, \tilde{R})$ where A/R

pts $\leftrightsquigarrow$
$$\begin{tikzcd} K & A \arrow[l] & \\ K^+ \arrow[u, hook] & \tilde{R} \arrow[l] \arrow[u, hook] & \\ & & R \arrow[ul] \arrow[ull, bend left] \end{tikzcd}$$

Conversely
$$\begin{tikzcd} K & A \arrow[l] & \\ K^+ \arrow[u, hook] & \tilde{R} \arrow[l, dashed, "\exists \text{ by}"', "\text{integrality}"] \arrow[u, hook] & \\ & & R \arrow[ul] \arrow[ull, bend left] \end{tikzcd}$$

So pts =
$$\begin{tikzcd} K & A \arrow[l] \\ K^+ \arrow[u, hook] & R \arrow[l] \arrow[u, hook] \end{tikzcd} \leftrightarrow \begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \operatorname{Spec} A \arrow[d] \\ \operatorname{Spec} K^+ \arrow[r] & \operatorname{Spec} R \end{tikzcd}$$

Get Fully Faithful Functors

$$\mathrm{Sch} \overset{ad}{\underset{ad/\mathbb{Z}}{\rightrightarrows}} (\text{Disc Adic Spaces})$$

$$\mathrm{Sch}/R \overset{ad}{\underset{ad/R}{\rightrightarrows}} (\text{Disc Adic Spaces})/\mathrm{Spa}(R,R)$$

Affine Locally

$$X = \mathrm{Spec}\,A \longmapsto X^{ad} = \mathrm{Spa}(A,A)$$

$$X = \mathrm{Spec}\,A \longmapsto X^{ad/\mathbb{Z}} = \mathrm{Spa}(A,\tilde{\mathbb{Z}})$$

$$\begin{array}{c} X = \mathrm{Spec}\,A \\ \downarrow \\ \mathrm{Spec}\,R \end{array} \longmapsto X^{ad/R} = \mathrm{Spa}(A,\tilde{R})$$

Points

$$X^{ad} \longleftrightarrow \left(\mathrm{Spec}\,k^{+} \longrightarrow X\right)/\sim$$

v.R.

$$X^{ad/R} \longleftrightarrow \left(\begin{tikzcd} \mathrm{Spec}\,k \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}\,k^{+} \arrow[r] & \mathrm{Spec}\,R \end{tikzcd}\right)/\sim$$

Notice have comparison maps $X^{ad} \longrightarrow X^{ad/\mathbb{Z}}$

<u>Indeed</u>

$$\mathrm{Spa}(A,A) \longrightarrow \mathrm{Spa}(A,\tilde{\mathbb{Z}})$$
$$v_x \longmapsto v_x$$

& fact that $|A|_x \leq 1 \Rightarrow |\tilde{\mathbb{Z}}|_x \leq 1$.

<u>Similarly</u> if X/R then $X^{ad} \longrightarrow X^{ad/R}$.

<u>Note:</u> In terms of pts

$(\mathrm{Spec}\, K^+ \longrightarrow X)$

$\updownarrow$

$$\left(\begin{tikzcd} \mathrm{Spec}\, K \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}\, K^+ \arrow[r] \arrow[ur] & R \end{tikzcd}\right)$$

$(*)$

<u>Prop</u> If X/R is sep finite type then

$$X^{ad} \longrightarrow X^{ad/R} \quad \text{is an open immersion}$$

which is an iso ~~[illegible]~~ if X/R is proper.

<u>proof</u>

Check open immersion locally on source, reduce to $X = \mathrm{Spec}\, A$.

$$\mathrm{Spa}(A,A) \longrightarrow \mathrm{Spa}(A,\tilde{R})$$
$$\|$$
$$\bigcap_{f \in A} U\left(\frac{f}{1}\right) \qquad \checkmark$$

~~[illegible]~~ If X is proper $x \in X^{ad/R}$

$$\begin{tikzcd} \mathrm{Spec}\, K \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}\, K^+ \arrow[r] \arrow[ur, dashed, "\exists !"] & \mathrm{Spec}\, R \end{tikzcd} \rightsquigarrow \begin{array}{c} (\mathrm{Spec}\, K^+ \longrightarrow X) \\ \uparrow \\ X^{ad} \end{array}$$

induces x w.r.t. $(*)$.

ϕ

Reorient Ourselves

→ Given some Huber pair (A, A^+) we get an abelian category $\mathrm{Mod}((A,A^+)_\square)$

⇝ We'd like to glue this into some category of quasi Coherent $(\mathcal{O}_X, \mathcal{O}_X^+)_\square$ modules on an adic space X. (i.e. Sheaves of solid modules).

⚠ Localization is not flat!

Example $A = \mathbb{Z}[T]$, $A_\infty = \mathbb{Z}((T^{-1}))$

$$X = \mathrm{Spa}(A, \mathbb{Z}), \quad U = \mathrm{Spa}(A, A) = \{T \leq 1\} = U\left(\tfrac{T}{1}\right)$$

Get restriction map

$$\begin{array}{ccc} \mathrm{Mod}((\mathcal{O}_X, \mathcal{O}_X^+)_\square) & \longrightarrow & \mathrm{Mod}((\mathcal{O}_U, \mathcal{O}_U^+)_\square) \\ \| & & \| \\ \mathrm{Mod}((A,\mathbb{Z})_\square) & \xrightarrow[- \otimes_{(A,\mathbb{Z})_\square} A_\square]{} & \mathrm{Mod}((A,A)_\square) = \mathrm{Mod}(A_\square) \end{array}$$

__Notice__

$$\phi : A \longrightarrow A_\infty \quad \rightsquigarrow \quad \phi_{|U} : \begin{array}{ccc} A \otimes_{(A,\mathbb{Z})_\square} A_\square & \longrightarrow & A_\infty \otimes_{(A,\mathbb{Z})_\square} A_\square \\ \| & & \| \\ A & \xrightarrow{0} & 0 \end{array}$$

injective