

Globalization of Solid Modules & Coherent Cohomology w/ Compact Support

①

§1: Relativizing Theorems (1)-(3) from last week.

Defn

Let $R \rightarrow A$ be a map of finitely generated \mathbb{Z} -algebras. We define $(A, R)_{\square}$ to be the pre-analytic ring whose underlying condensed ring is A (as a discrete ring), &

$$(A, R)_{\square}[S] := R_{\square}[S] \otimes_R A$$

$\uparrow = \varprojlim R[S_i]$

Remark

$$R_{\square}[S] \otimes_R^L A \simeq R_{\square}[S] \otimes_R A \quad \text{see by viewing}$$

$$R_{\square}[S] \simeq \prod_I R$$

Theorem 0

$(A, R)_{\square}$ is analytic.

Rephrase the theorems from last time in the relative setting.

Note

The initial reduction ~~from~~^{to} the case $A = \mathbb{Z}[t]$ from last time can be done in the relative setting, to $A = R[t]$, & even similar proofs go through.

Theorem R1

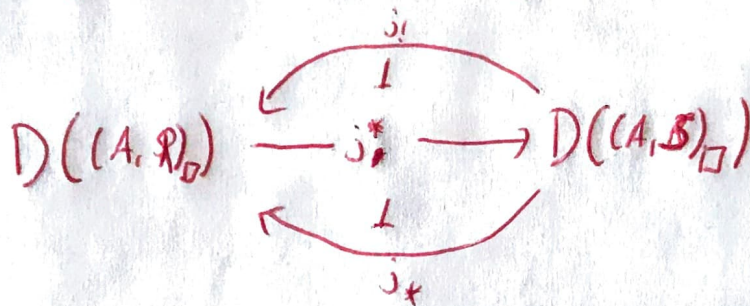
(2)

Let $R \rightarrow S \rightarrow A$ be maps of finitely generated \mathbb{Z} -algs.

(R1.1) $j_* : D((A, S)_{\square}) \leftarrow D((A, R)_{\square})$ forgetful admits
a left adjoint

$j^* := - \otimes_{(A, R)_{\square}}^L (A, S)_{\square}$ which in turn admits

a fully faithful left adjoint $j!$



(R1.2) Projection Formula: $M \in D((A, R)_{\square})$ -module

$$j! j^*(M) \simeq M \otimes_{(A, R)_{\square}}^L j! A$$

↑ this is enough!

Defⁿ (Assuming Theorem R1) can define relative
compactly supported cohomology:

$$j : \text{Spec } A \longrightarrow \text{Spec } R \quad \text{s.t. } \mathbb{Z}$$

$$D(A_{\square}) = D((A, A)_{\square}) \xrightarrow{j!} D((A, R)_{\square}) \xrightarrow{\text{Forget}} D(R_{\square})$$

$j_!$ ↗

Theorem R2

(3)

(R2.3) $S_!$ commutes w/ direct sums & satisfies a projective formula:

$$M \in D(R_{\square}), \quad N \in D(A_{\square})$$

$$M \otimes_{R_{\square}}^L S_! N \simeq S_! \left((M \otimes_{R_{\square}}^L A_{\square}) \otimes_{A_{\square}}^L N \right)$$

If S has finite Tor-dimension even

(R2.1) $S_!$ admits a right adjoint

$$S^! = \text{RHom}_{(A,R)_{\square}}(-, \mathcal{Y} \otimes_{(A,R)_{\square}}^L A_{\square})$$

(R2.2) $S_!$ preserves compact objects

Theorem R3 (If S has finite Tor-dimension)

(R3.1) $S^!(R) \in D(A_{\square})$ is discrete & bounded to the left complex of S -genl A -modules. If S has finite Tor dimension ~~un~~ $S^!R$ is bounded.

(R3.2) If S has finite Tor-dimension, $S^!$ is given by

$$S^!(M) = (M \otimes_{R_{\square}}^L A_{\square}) \otimes_{A_{\square}}^L S^!R$$

(R3.3) If \mathcal{S} has finite Tor dimension $\mathcal{S}!$ preserves discrete $\textcircled{4}$ objects.

(R3.4) If \mathcal{S} is a complete intersection, $\mathcal{S}!(R)$ is invertible.

Theorem (R4)

$$\text{Spec } \mathbb{A} \xrightarrow{f} \text{Spec } \mathbb{B} \xrightarrow{g} \text{Spec } \mathbb{R} \quad \text{s.t. } \mathbb{Z}$$

$$(g \circ f)_! \simeq g_! \circ f_!$$

$$(g \circ f)^! \simeq f^! \circ g^!$$

A word on R4

$$\begin{array}{ccccc}
 D(\mathbb{A}_{\mathbb{Q}}) & \xrightarrow{\quad} & D(\mathbb{A}, \mathcal{S})_{\mathbb{Q}} & \xrightarrow{j_!} & D(\mathbb{A}, \mathbb{R})_{\mathbb{Q}} \\
 \searrow f_! & & \downarrow \text{Forget} \textcircled{1} & & \downarrow \text{Forget} \\
 & & D(\mathbb{S}_{\mathbb{Q}}) & \xrightarrow{\hat{j}_!} & D(\mathbb{S}, \mathbb{R})_{\mathbb{Q}} \\
 & & \searrow g_! & & \downarrow \text{Forget} \\
 & & & & D(\mathbb{R}_{\mathbb{Q}})
 \end{array}$$

$(g \circ f)_!$

Subsies to show $\textcircled{1}$ commutes

~~But~~ ~~!~~ Certainly the replacing $j_!$ w/ j_* . Is that enough since adjointing is formal?

§2: Globalization.

Want: A geometric framework so that $f: X \rightarrow Y$ get

$$\text{some } f_! : \underbrace{D(X_{\square})}_{\uparrow \text{ what is this?}} \longrightarrow D(Y_{\square})$$

Know how to do this when (X, \mathcal{O}_X) corresponds to (pairs of) \mathcal{O} -gen'd \mathbb{Z} -algebras. ~~not~~

Lemma $R \xrightarrow{\phi} A$ map of \mathcal{O} -gen'd \mathbb{Z} -alg's.

$$A^+ = \widetilde{\phi(R)} = \text{Integral closure of } \phi(R) \text{ in } A.$$

Then the analytic rings

$$(A, R) \xleftarrow{\sim} (A, A^+)$$

PS/

$$R \xrightarrow{\text{finte}} A^+ \longrightarrow A \quad \underline{A^+ \text{ a finite } R\text{-module}}$$

$$\begin{aligned}
(A, R)_{\square} [S] &= R_{\square} [S] \otimes_R A \\
&= \underbrace{R_{\square} [S] \otimes_R A^+}_{\substack{\text{SI finite} \\ \downarrow}} \otimes_{A^+} A \approx A^+ [S] \otimes_{A^+} A \\
&= \left(\prod_{\mathbb{I}} R \right) \otimes_R A^+ \approx (A, A^+)_{\square} [S] \\
&\quad \downarrow \\
&\quad \prod_{\mathbb{I}} A^+
\end{aligned}$$

\emptyset

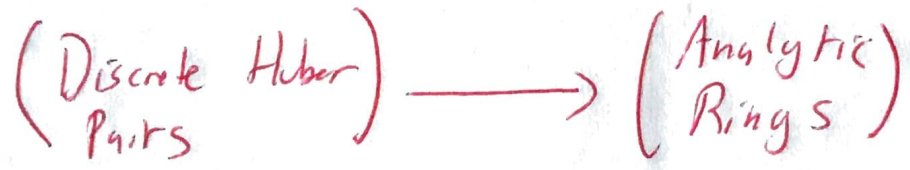
So our functor $(R \rightarrow A) \mapsto \mathcal{D}(A, R)_0$ is most naturally sourced in the category of pairs (A, A^+) w/ A s.g. / \mathbb{Z} & A^+ a s.g. int. closed subalg.

This naturally puts us in the setting of Huber's Theory.

Defⁿ

A discrete Huber Pair is a pair (A, A^+) of discrete rings w/ $A^+ \subseteq A$ integrally closed.

Get a functor



$$(A, A^+) \longmapsto (A, A^+)_{\square} := \varinjlim_{(B, B^+) \rightarrow (A, A^+} (B, B^+)_{\square}$$

\uparrow not s.g. int... \uparrow s.g. int DHP

Remark:

Huber pairs exist more generally for topological rings.
 A = topological ring w/ $A_0 \subseteq A$ open & I-adic for $I \subseteq A_0$.
 A^+ = open integrally closed \wedge subring bundle.

Naturally includes adic rings & their generic fibers.

Defⁿ Given a Discrete Huber pair (A, A^+)
one can form

$$\text{Spa}(A, A^+) := \{ | \cdot | : A \rightarrow \Gamma \cup \{0\} \mid |A^+| \leq 1 \} / \cong$$

(w/ Γ a totally ordered abelian group (of elts > 0)
 $| \cdot |$ nonarchimedean absolute value
 $|0| = 0, |1| = 1, |xy| = |x| \cdot |y|, |x+y| \leq \max\{|x|, |y|\}$.)

Notation Convention

$x \in \text{Spa}(A, A^+)$ is a valuation on A .

We write instead of $x(\mathcal{O}) = : | \mathcal{O}(x) |$ ← This is often how we get points of Spa

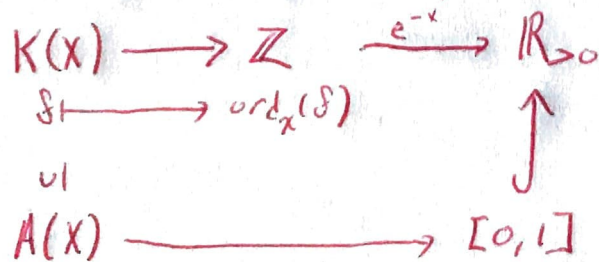
Remarks

① More general Huber rings we require $| \cdot |$ to be cts

② Why should valuations be points?

ⓐ Peter's universal compactification argument.

ⓑ $X = \text{curve}$. $x \in X$ a point, get a valuation on



Recover $A(X)$ (thus X) from the ^{correct} collection of vals on $K(X)$

© Points of $\overline{D}^1 = \text{Spa}(K\langle X \rangle, \mathcal{O}_K\langle X \rangle)$ K nonarch field.
 Should be an analytic space whose points are functions
 are convergent power series.

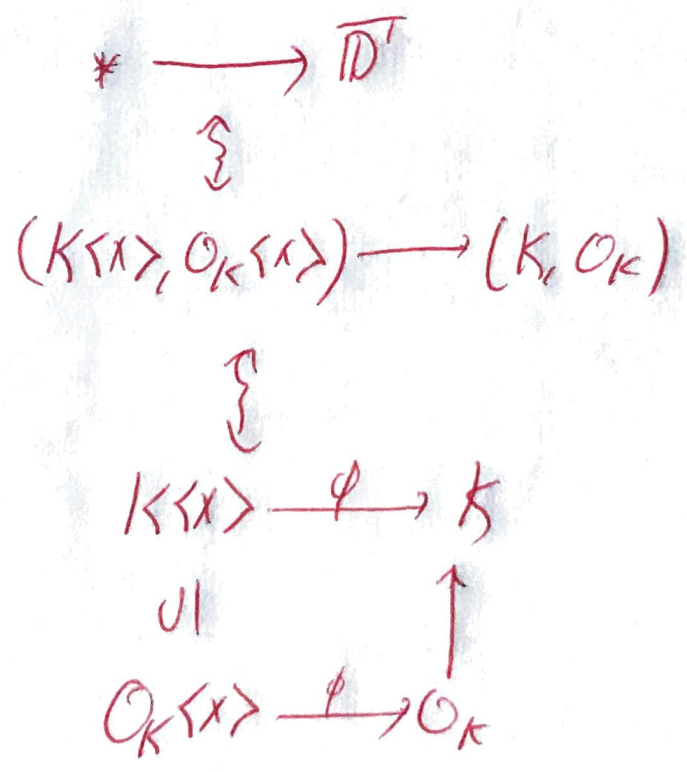
E.g. $x \in \mathcal{O}_K$. $f \in K\langle X \rangle$ can be evaluated @ x .
 $f(x) \in K$ & $|f(x)| \in \mathbb{R}$ And indep

$f \mapsto |f(x)|$ is a valuation of $K\langle X \rangle$
 which is sl on $\mathcal{O}_K\langle X \rangle \implies x \in \overline{D}^1$.

① K -points of \overline{D}^1 .

(K, \mathcal{O}_K) -points:

$* = \text{Spa}(K, \mathcal{O}_K)$



Determined by $\phi(x) \in \mathcal{O}_K$. (by continuity).

so $\overline{D}^1(K, \mathcal{O}_K) = \mathcal{O}_K$.

Topologizing $\text{Spa}(A, A^+) = X$

(9)

Open Sets Can Be Defined By Inequalities

Basis given by

$$U\left(\frac{g_1, \dots, g_n}{f}\right) = \left\{ x \in X \mid |g_i(x)| \leq |f(x)| \neq 0 \right\}$$

as $g_1, \dots, g_n, f \in A$ vary.

This makes X a spectral space ($\cong \text{Spec}$ some ring)

Notice

$$\text{Spa}(A, A^+) \subseteq \text{Spa}(A, \tilde{\mathbb{Z}}) =: \text{Spv}(A)$$

↑
open.

Sp

↑ All valuations

Since $|\tilde{\mathbb{Z}}| \leq 1$ by star ultrametric.

Prop

$$\left\{ \begin{array}{l} \text{Integrally} \\ \text{closed} \\ A^+ = A \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Open } U = \text{Spa}(A, \tilde{\mathbb{Z}}) \\ \text{s.t.} \\ U = \bigcap_{f \in A^+} U\left(\frac{f}{1}\right) \end{array} \right\}$$

$$A \longmapsto U = \text{Spa}(A, A^+) = \bigcap_{f \in A^+} U\left(\frac{f}{1}\right)$$

$$A^+ := \left\{ f \in A \mid \begin{array}{l} \forall x \in U \\ |f(x)| \leq 1 \end{array} \right\}$$

$$\longleftarrow U$$

Proof

(11)

1) $A^t(U)$ is integrally closed. Indeed, if $f \in A^t(U)$ int $(A^t(U))$

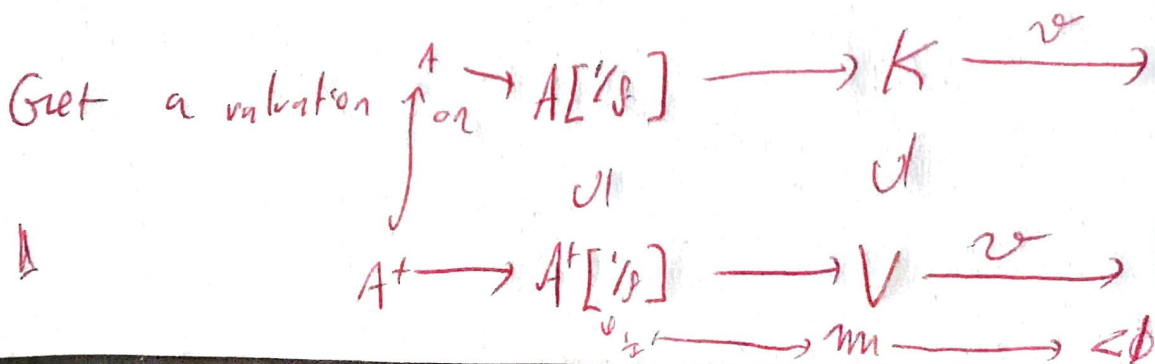
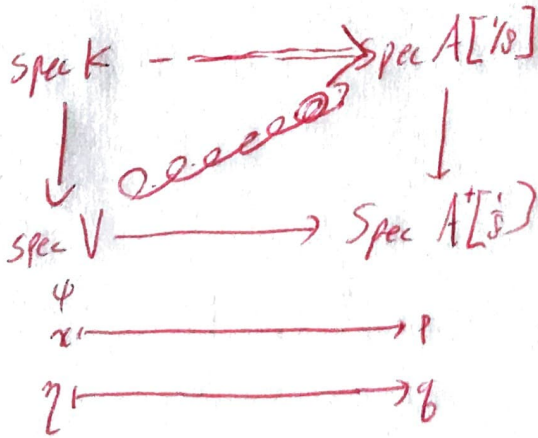
$\Rightarrow \|D(x)\| \leq 1 \quad \forall x \in U$ by arithmetic $\Rightarrow f \in A^t(U)$

2) Take $U = \bigcap_{S \in T} U(\frac{S}{I})$ on RHS. $A^t = \overline{\langle S \mid S \in I \rangle} \subseteq A$

$\Rightarrow \text{Spec}(A, A^t) = U$

3) $A^t(\text{Spec}(A, A^t)) = \{S \in A \mid \forall x \in \text{Spec}(A, A^t), \|D(x)\| \leq 1\} \stackrel{\cong}{=} A^t$

$S \notin A^t \Rightarrow S \notin A^t[\frac{1}{S}] \subseteq A[\frac{1}{S}]$
 \cup
 $\frac{1}{S} \in P \uparrow$ prime
 \cup
 $q \in$ minimal prime.



$\Rightarrow \|D\| \geq 1$
 \nexists

Equivalence of Categories

~~Affinoid~~

$$\left(\begin{array}{c} \text{Disc Huber} \\ \text{Pairs} \end{array} \right) \xrightarrow{\text{Spa}} \left(\begin{array}{cc} \text{Affinoid} & \text{Discrete} \\ \text{alge} & \text{Spaces} \end{array} \right)$$

$$(A, A^+) \longmapsto \text{Spa}(A, A^+)$$

$$(A, A^+) \rightarrow (B, B^+) \longmapsto \begin{array}{c} \text{Spa}(A, A^+) \\ \uparrow \\ \text{Spa}(B, B^+) \end{array}$$

$$\begin{array}{ccccc} A^+ & \rightarrow & B^+ & \rightarrow & \Gamma_{<1} \\ \eta & & \eta & & \eta \\ A & \rightarrow & B & \rightarrow & \Gamma \\ & & \uparrow & & \\ & & \mathcal{D} & : & B \rightarrow \Gamma \\ & & \cup & & \cup \\ & & B^+ & \rightarrow & \Gamma_{<1} \end{array}$$

Recall

For Schemes: $X = \text{Spec } A$

$$D(\mathcal{D}) = \cup = \text{Spec}(A_{\mathcal{D}})$$

What does (topological) localization for adic spaces look like algebraically?

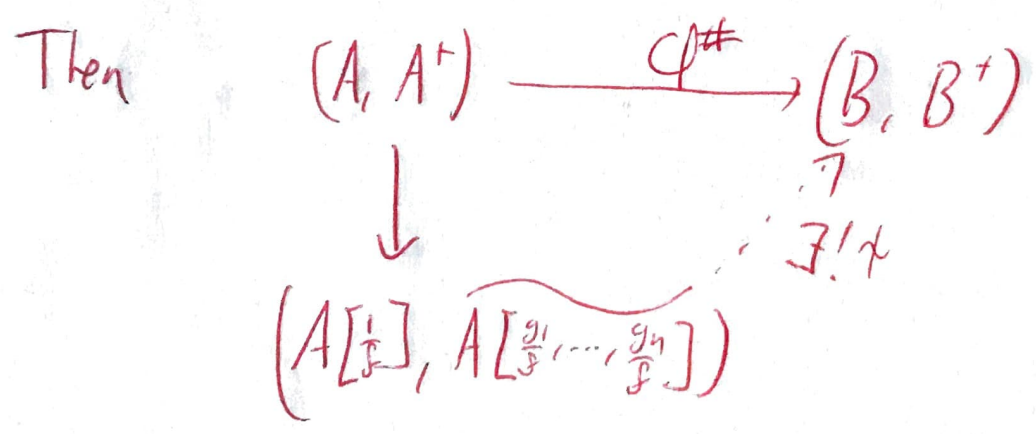
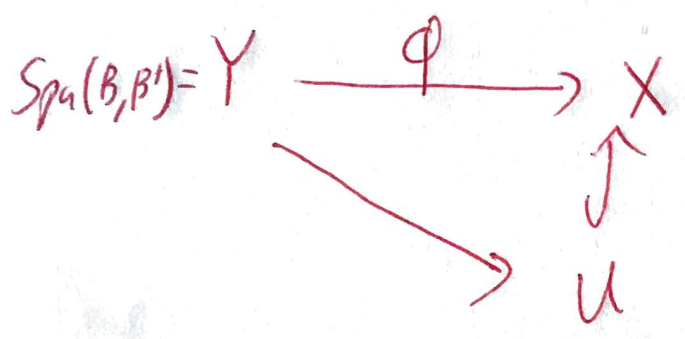
Prop: $X = \text{Sp}_n(A, A^+)$

$U = U\left(\frac{g_1, \dots, g_n}{f}\right) = \left\{ x \in X \mid |g_i(x)| \leq |f(x)| \neq 0 \right\}$

then $U \cong \text{Sp}_n\left(A\left[\frac{1}{f}\right], \widetilde{A^+\left[\frac{g_1}{f}, \dots, \frac{g_n}{f}\right]}\right)$

precisely

Suppose $(A, A^+) \rightarrow (B, B^+)$ induces



& $\text{Sp}_n\left((A\left[\frac{1}{f}\right], \widetilde{A\left[\frac{g_1}{f}, \dots, \frac{g_n}{f}\right]})\right) \rightarrow X$ is a homeomorphism onto U .

Proof

(14)

First notice $\phi^*(S) \in B^*$.

Else, $\exists \phi^*(S) \in \mathfrak{p} \in \text{Spec } B$

$$\begin{array}{ccc} & \xrightarrow{\mathcal{V}_{\mathfrak{p}}} & \\ B & \longrightarrow & \text{Frac}(B/\mathfrak{p}) \xrightarrow{\text{triv}} \{0, 1\} \\ b \mapsto & & \begin{cases} 0 & b \in \mathfrak{p} \\ 1 & \text{else} \end{cases} \end{array}$$

$$\& \mathcal{V}_{\mathfrak{p}}(\phi^*(S)) = 0$$

$$|\phi^*(S)(\mathfrak{p})| = |S(\phi(\mathfrak{p}))| \neq 0 \quad \downarrow$$

$\uparrow \in U$

But $\forall y \in Y$

$$|\phi^*(S)(y)| = |S(\phi(y))| \neq 0$$

$$\begin{array}{ccc} \text{So} & A & \xrightarrow{\phi^*} B \\ & \searrow & \nearrow \\ & A[\frac{1}{S}] & \xrightarrow{\phi^*} \end{array}$$

Now $\forall y \in Y$

$$|\phi^*(\frac{g_i}{f})(y)| = |(\frac{g_i}{f})(\phi(y))| \leq 1$$

$\uparrow \in U$

$$\Rightarrow \phi^*\left(A^*\left[\frac{g_1}{f}, \dots, \frac{g_n}{f}\right]\right) \subseteq B^+$$

In particular, we see that the open set

$U = U\left(\frac{g_1, \dots, g_n}{f}\right)$ uniquely determines the discrete Huber pair $(A[1/f], A^+[\frac{g_1}{f}, \dots, \frac{g_n}{f}])$.

Defⁿ $X = \text{Spa}(A, A^+)$.

Define sheaves $\mathcal{O}_X, \mathcal{O}_X^+$ on X by defining them on a basis opens of the form $U = U\left(\frac{g_1, \dots, g_n}{f}\right)$

$$\mathcal{O}_X(U) = A[1/f]$$

$$\mathcal{O}_X^+(U) = A^+[\frac{g_1}{f}, \dots, \frac{g_n}{f}].$$

Propⁿ

- 1) \mathcal{O}_X & \mathcal{O}_X^+ are sheaves.
- 2) $x \in X$, extends uniquely to a valuation on $\mathcal{O}_{X, x}$.
- 3) $\mathcal{O}_X^+(U) = \left\{ f \in \mathcal{O}_X(U) \mid |f(x)| \leq 1 \ \forall x \in U \right\}$
↑ makes sense on stalks

Proof

Let $\varphi: \text{Spec } A \longrightarrow \text{Spa}(A, A^+)$

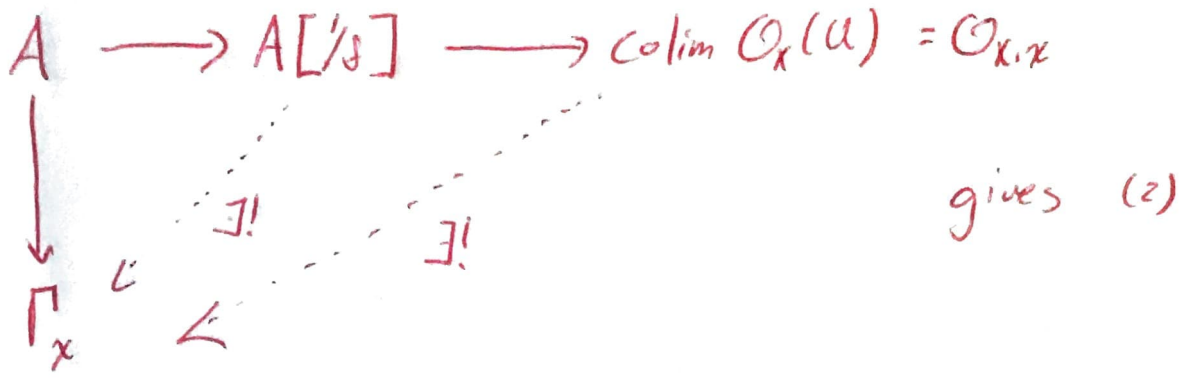
$$p \longmapsto \mathcal{O}_p: A \longrightarrow \text{Frac}(A/p) \xrightarrow{\text{inv}} \{0, 1\}$$

This is continuous, & $\varphi^{-1}\left(U\left(\frac{g_1, \dots, g_n}{f}\right)\right) = D(f)$

$$\text{So } \mathcal{O}_X(U) = A[1/f] = \mathcal{O}_{\text{Spec } A}(D(f)) = \mathcal{O}_{\text{Spec } A}(\varphi^{-1}(U))$$

$\implies \mathcal{O}_X = \varphi_* \mathcal{O}_{\text{Spec } A}$ & thus a sheaf!

2)



3) True on $(A[\frac{1}{S}], A'[\frac{y_1}{S}, \dots, \frac{y_n}{S}])$ (already saw).

Back to

1) $\mathcal{O}_X^+ \subseteq \mathcal{O}_X$ now given by a stalkwise condition.

So it's a sheaf!

(give uniquely in \mathcal{O}_X then check on stalks) \emptyset

Defn

A discrete adic space is a triple $(X, \mathcal{O}_X, (1 \cdot (x))_{x \in X})$ where X a top space, \mathcal{O}_X a sheaf of rings, $1 \cdot (x)$ a valuation on the stalk $\mathcal{O}_{X,x}$, which is locally $(\text{Spa}(A, A'), \mathcal{O}_{\text{Spa}(A, A')}, (1 \cdot (x))_{x \in \text{Spa}(A, A')})$ ($\mathcal{O}_{X,x}$) a valuation equiv. class

$$(\text{Spa}(A, A'), \mathcal{O}_{\text{Spa}(A, A')}, (1 \cdot (x))_{x \in \text{Spa}(A, A')})$$

for (A, A') a discrete Huber pair.

Rmk * A ~~morphism~~ (pre)-adic space is $(X, \mathcal{O}_X, (\mathcal{V}_x)_{x \in X})$

s.t. $\mathcal{O}_{X,x}$ is local w/ max' ideal $\text{supp}(\mathcal{V}_x)$.

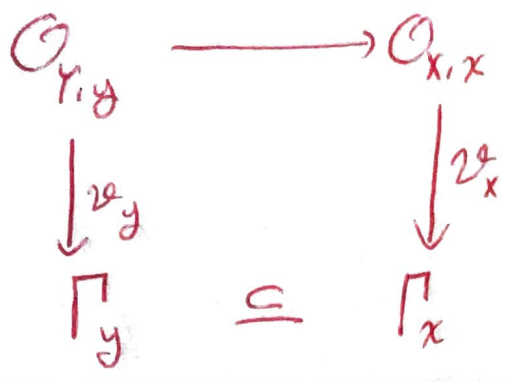
* A morphism $\mathcal{S} = (X, \mathcal{O}_X, (\mathcal{V}_x)_{x \in X}) \longrightarrow (Y, \mathcal{O}_Y, (\mathcal{V}_y)_{y \in Y})$ is ...

of pre-adic spaces

$$\left. \begin{aligned} f: X &\longrightarrow Y \\ f^*: \mathcal{O}_Y &\longrightarrow f_*\mathcal{O}_X \end{aligned} \right\} \text{map of LRS.}$$

So that

$$f(x) = y$$



Given (X, \mathcal{O}_X) ($\varphi_x = \text{loc}(x)$)

$$\mathcal{O}_x^+ \cong \mathcal{U} \longmapsto \{f \in \mathcal{O}_x(\mathcal{U}) \mid |f(x)| \leq 1 \ \forall x \in \mathcal{U}\}$$

Then $\mathcal{O}_{x,x}^+ = \{f \in \mathcal{O}_{x,x} \mid |f(x)| \leq 1\}$ is a valuation ring w/ max ideal $\mathcal{M}_{x,x} = \{f \in \mathcal{O}_{x,x} \mid |f(x)| < 1\}$

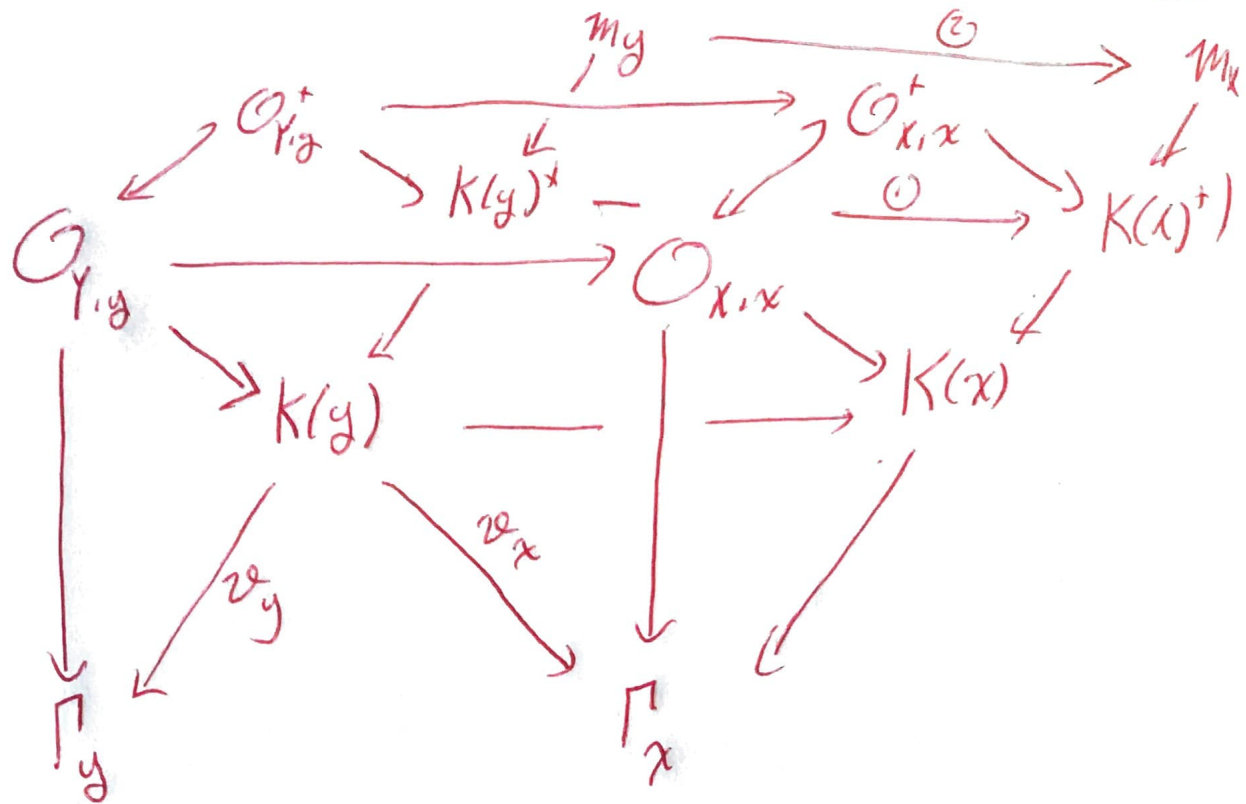
Prop $f: (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ map of locally ringed spaces is a map of (pre)-adic spaces iff

$$1) \begin{array}{ccc} \mathcal{O}_Y & \longrightarrow & f_*\mathcal{O}_X \\ \uparrow & & \uparrow \\ \mathcal{O}_Y^+ & \longrightarrow & f_*\mathcal{O}_X^+ \end{array} \quad \& \quad 2) \begin{array}{ccc} \mathcal{O}_Y^+ & \longrightarrow & f_*\mathcal{O}_Y^+ \\ & & \text{is a local map.} \end{array}$$

In particular, if $X = \text{Sp}(A, A^+)$ & $Y = \text{Sp}(B, B^+)$

$$\text{Hom}(X, Y) = \text{Hom}((B, B^+), (A, A^+))$$

More generally, \forall adicoid $\text{Hom}(X, Y) = \text{Hom}(\text{Hom}(\mathcal{O}_Y, \mathcal{O}_X), \text{Hom}(\mathcal{O}_X, \mathcal{O}_Y))$ $\sim \text{Hom}(\mathcal{O}_Y, \mathcal{O}_X)$



Goal valuation rings for $v_y \in K(y)^+$ & $v_x \in A$ in $K(y)$

agree.

- (1) $|S(y)| \leq |g(y)| \xRightarrow{1} |S(x)| \leq |g(x)| \xRightarrow{1} K(y)^+ = A$
- (2) $|S(y)| < |g(y)| \xRightarrow{1} |S(x)| < |g(x)| \xRightarrow{1}$

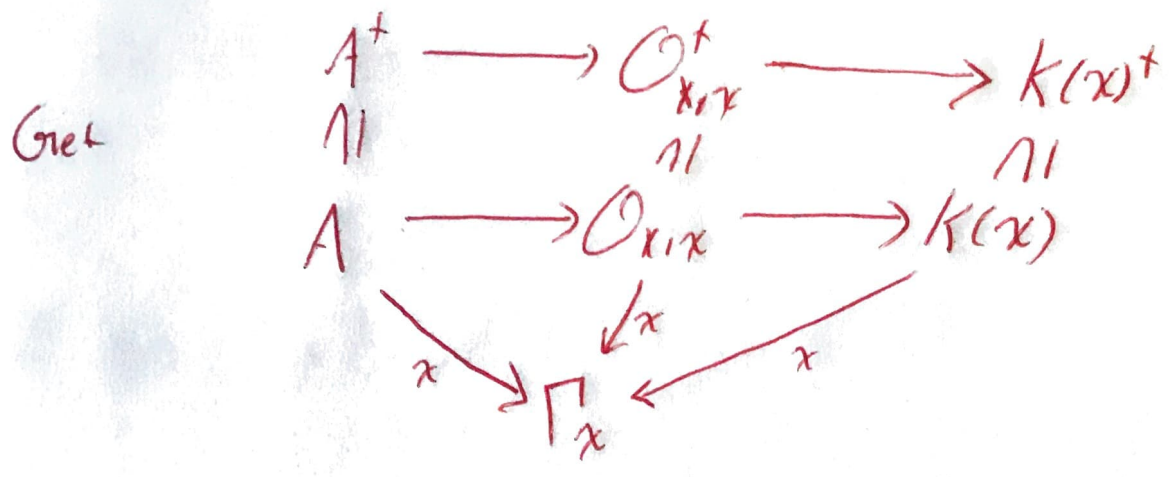
(2) $\exists s \notin K(y)^+ \Rightarrow |S(x)| > |g(x)|$
 $\Rightarrow |S(x)| > |g(x)|$
 $\Rightarrow |S(x)| \notin A$

so $A \in K(y)^+$



Points on adic spaces

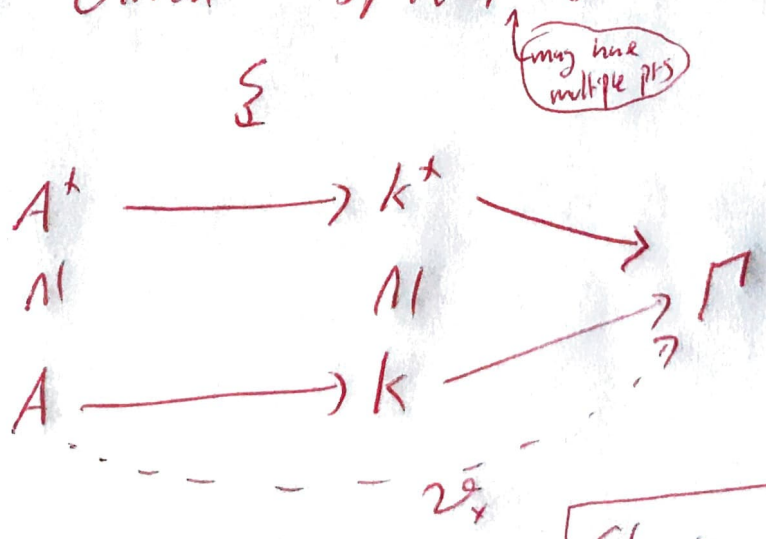
Given $x \in X = \text{Spa}(A, A^+)$



So get $(A, A^+) \longrightarrow (k(x), k(x)^+)$

i.e. $\text{Spa}(k(x), k(x)^+) \longrightarrow X$

Conversely Given $\text{Spa}(k, k^+) \longrightarrow X$



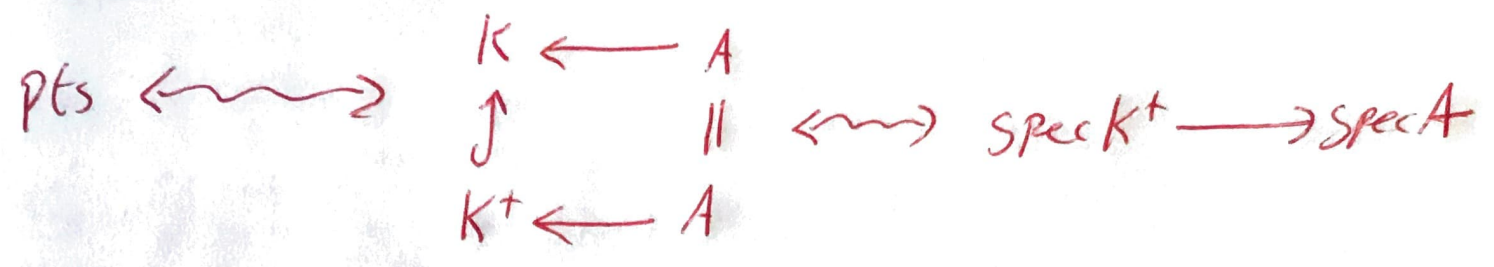
Choosing \mathbb{A} & choosing a valuation
 $VR =$ a valuation

Get $x \in X$

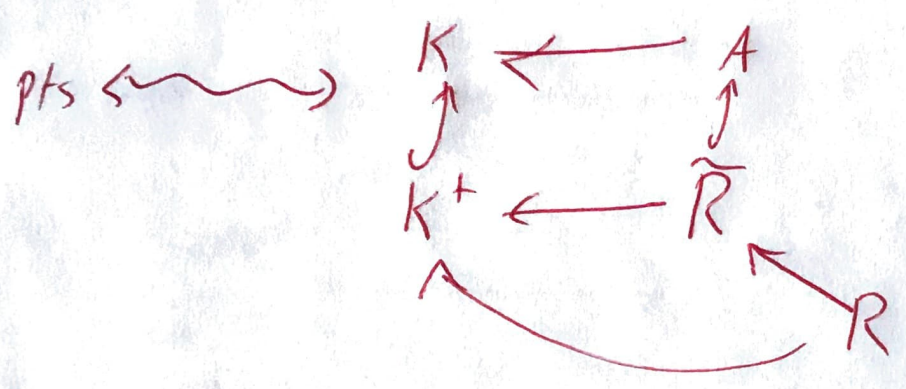
Slogan
 Points = (maps from $\text{Spa}(k, k^+)$ where k^+ is a valuation ring)

2 Special Cases

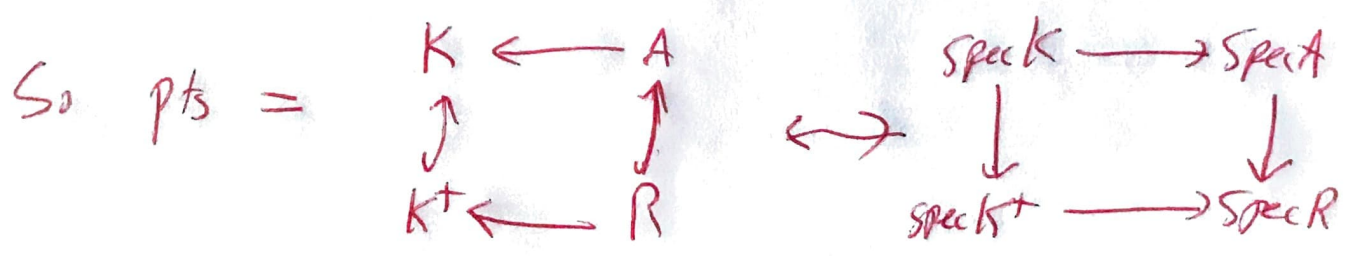
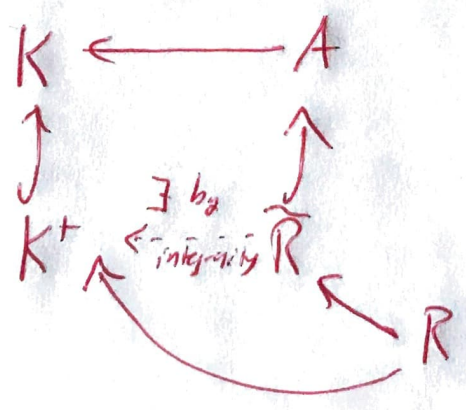
① $X = \text{Spa}(A, A)$



② $X = \text{Spa}(A, \tilde{R})$ where A/R



Conversely



Get Fully Faithful Functors

$$\text{Sch} \begin{array}{c} \xrightarrow{\text{ad}} \\ \xrightarrow{\text{ad}/\mathbb{Z}} \end{array} \rightarrow (\text{Disc Adic Spaces})$$

$$\text{Sch}/R \begin{array}{c} \xrightarrow{\text{ad}} \\ \xrightarrow{\text{ad}/R} \end{array} \rightarrow (\text{Disc Adic Spaces})/\text{Spec}(R, R)$$

Adic Locality

$$X = \text{Spec } A \longmapsto X^{\text{ad}} = \text{Spa}(A, A)$$

$$X = \text{Spec } A \longmapsto X^{\text{ad}/\mathbb{Z}} = \text{Spa}(A, \tilde{\mathbb{Z}})$$

$$\begin{array}{ccc} X = \text{Spec } A & \longmapsto & X^{\text{ad}/R} = \text{Spa}(A, \tilde{R}) \\ \downarrow & & \\ \text{Spec } R & & \end{array}$$

Points

$$X^{\text{ad}} \longleftrightarrow \left(\text{Spec } k^+ \xrightarrow{\text{v.R.}} X \right) / \sim$$

$$X^{\text{ad}/R} \longleftrightarrow \left(\begin{array}{ccc} \text{Spec } k & \longrightarrow & X \\ \downarrow & & \downarrow \\ \text{Spec } k^+ & \longrightarrow & \text{Spec } R \end{array} \right) / \sim$$

Notice how comparison maps $X^{qd} \rightarrow X^{qd/\mathbb{Z}}$

Indeed

$$\begin{array}{ccc} \text{Spa}(A, A) & \longrightarrow & \text{Spa}(A, \tilde{\mathbb{Z}}) \\ \mathcal{O}_x \downarrow & & \downarrow \mathcal{O}_x \end{array}$$

Note: In terms of pts
($\text{Spec } k^+ \rightarrow X$)
 \downarrow
 $\text{Spec } k \rightarrow X$
 \downarrow
 $\text{Spec } k^+ \rightarrow R$
 (*)

Fact that $|A|_x \leq 1 \implies |\tilde{\mathbb{Z}}|_x \leq 1$.

Similarly if X/R then $X^{qd} \rightarrow X^{qd/R}$.

Prop If X/R is ^{slp} finite type then

$$X^{qd} \longrightarrow X^{qd/R} \text{ is an open immersion}$$

which is an iso ~~if~~ X/R is proper.

proof

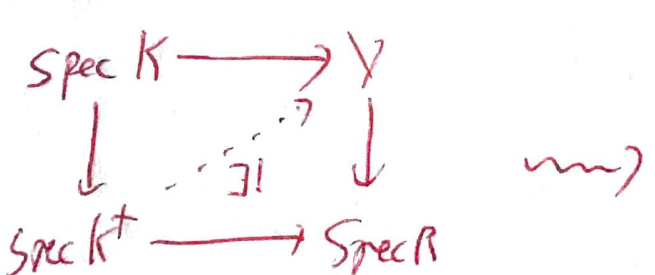
Check open immersion locally on source, reduce to $X = \text{Spec } A$.

$$\text{Spa}(A, A) \longrightarrow \text{Spa}(A, \tilde{R})$$

$$\bigcap_{f \in A} U\left(\frac{f}{1}\right)$$



Moreover If X is proper, $x \in X^{qd/R}$,



$$(\text{Spec } k^+ \rightarrow X)$$

X^{qd} induces x via (*).



Reorient Ourselves

(23)

→ Given some Huber pair (A, A^+) we get an abelian category $\text{Mod}((A, A^+)_{\square})$

→ We'd like to glue this into some category of quasi-coherent $(\mathcal{O}_X, \mathcal{O}_X^+)_{\square}$ modules on an adic space X . (i.e. sheaves of solid modules).

⚠ Localization is not flat!

Example $A = \mathbb{Z}[T], A_{\infty} = \mathbb{Z}\langle T^{-1} \rangle$

$$X = \text{Spa}(A, \mathbb{Z}), U = \text{Spa}(A, A) = \{T \neq 0\} = U\left(\frac{T}{1}\right)$$

Get restriction map

$$\text{Mod}((\mathcal{O}_X, \mathcal{O}_X^+)_{\square}) \longrightarrow \text{Mod}((\mathcal{O}_U, \mathcal{O}_U^+)_{\square})$$

$$\text{Mod}((A, \mathbb{Z})_{\square}) \longrightarrow \text{Mod}((A, A)_{\square}) = \text{Mod}(A_{\square})$$

$- \otimes_{(A, \mathbb{Z})_{\square}} A_{\square}$

Notice

$$\begin{array}{ccc} \phi: A \longrightarrow A_{\infty} & \rightsquigarrow & \phi|_U: A \otimes_{(A, \mathbb{Z})_{\square}} A_{\square} \longrightarrow A_{\infty} \otimes_{(A, \mathbb{Z})_{\square}} A_{\square} \\ \text{injective} & & \begin{array}{ccc} \parallel & & \parallel \\ A & \xrightarrow{\circ} & 0 \end{array} \end{array}$$