Condensed Abelian Groups

→ Last time saw condensed sets gave a nice categorical framework to do topology.

→ This time see condensed abelian groups give a nice categorical framework to do topological algebra.

Example

Saw a glimpse of this last week.

$$0 \longrightarrow \mathbb{R}^{disc} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{R}^{disc} \longrightarrow 0$$

1) A location where $\mathbb{R}/\mathbb{R}^{disc}$ exists

2) As these are sheaves, evaluate @ points/opens to probe structure of $\mathbb{R}/\mathbb{R}^{disc}$ in terms of $H^1(-,\mathbb{R}^{disc}) \longrightarrow H^1(-,\mathbb{R})$

In particular, we benefit from having a robust setting in which to do homological algebra.

Recall Defined sites

$$\underset{\text{compact Hausdorff}}{\mathrm{CHaus}} \supseteq \underset{\text{profinite sets}}{\mathrm{PFSet}} \supseteq \underset{\text{extremally disconnected sets}}{\mathrm{EDSet}}$$

Covers = finite families of jointly surjective maps

$$\mathrm{Cond}(\mathrm{Ab}) = \mathrm{Ab}(\mathrm{CHaus}) = \mathrm{Ab}(\mathrm{PFSet}) = \mathrm{Ab}(\mathrm{EDSet}).$$

Main Theorem for Today

Cond(Ab) is an abelian category satisfying

(Ab3) Colimits exist

(Ab4) Coproducts are exact

(Ab5) Filtered Colimits are exact

(Ab6) J index, I_j family of filtered categories:

$$\operatorname*{colim}_{i_j \in \prod_j I_j} \prod_{j \in J} M_{i,j} \xrightarrow{\simeq} \prod_{j \in J} \operatorname*{colim}_{i \in I_j} M_{i,j} \quad \text{iso}$$

(Ab3*) Limits exist

(Ab4*) Products are exact.

(Symmetric Monoidal) $-\otimes-$ exists w/ relevant diagrams and unit.

(Closed) Internal Hom exists.

(Compact Projective Generators) $\overset{\text{in particular}}{\Longrightarrow}$ Enough Projectives

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$\boxed{\text{Def}^n}$ $\mathcal{A}$ an abelian category.

$M \in \mathcal{A}$ is

1) Compact if ~~[illegible]~~ $\mathrm{Hom}(M,-)$ commutes w/ filtered colimits

2) Projective if $\mathrm{Hom}(M,-)$ is exact.

Remarks

1. The category of abelian groups (more generally R-modules) satisfies the theorem.

2. Let $\mathbb{C}$ be a category, and $\mathrm{PreAb}(\mathbb{C})$ the category of presheaves of abelian groups. Then $\mathrm{PreAb}(\mathbb{C})$ satisfies the theorem.

   Pf/ Limits/Colimits/Injectivity/Surjectivity .... etc computed pointwise, so result follows formally from the case for Ab.

3. Let $\mathbb{C}$ be a (small) site. Then much of the theorem holds formally for $\mathrm{Ab}(\mathbb{C})$, but not all! In particular, $\mathrm{Ab}(\mathbb{C})$ is abelian & satisfies

   (AB3), (AB4), (AB5), (AB3*) (Symmetric Monoidal) (Closed).

   But not in general.

   (AB4*), (AB6), (Compact Projective Generators)

   It is these last three that make Cond(Ab) special!

Before moving into the condensed setting, discuss the formality of much of the theorem for $\mathrm{Ab}(\mathbb{C})$. It follow essentially from the ~~existence~~ result on ~~PreSh~~ $\mathrm{PreAb}(\mathbb{C})$ together w/ the exactness & adjointness properties of the sheafification functor $\#$.

$\boxed{\text{Example}}$ (AB3) for $Ab(\mathcal{C})$.

Let $\mathcal{F}_i$ be a system of sheaves. We

Claim $\underset{\substack{\uparrow \\ \text{presheaf colimit}}}{(\operatorname{colim}_i \mathcal{F}_i^{pre})}\overset{\#}{}$ satisfies colimit property

$\#$ = sheafification.

Indeed
$$\operatorname{Hom}_{Ab(\mathcal{C})}\left((\operatorname{colim} \mathcal{F}_i^{pre})^{\#}, \mathcal{G}\right) = \operatorname{Hom}_{PreAb(\mathcal{C})}\left(\operatorname{colim} \mathcal{F}_i^{pre}, \mathcal{G}^{pre}\right) \quad \text{(Adjunctions)}$$
$$= \lim_i \operatorname{Hom}_{PreAb(\mathcal{C})}\left(\mathcal{F}_i^{pre}, \mathcal{G}^{pre}\right)$$
$$= \lim \operatorname{Hom}_{Ab(\mathcal{C})}\left(\mathcal{F}_i, \mathcal{G}\right)$$

(AB4) $0 \to \mathcal{F}_i \to \mathcal{G}_i$ exact

$\Longrightarrow 0 \to \mathcal{F}_i(U) \to \mathcal{G}_i(U)$ exact (Γ left exact)

$\Longrightarrow 0 \to \coprod \mathcal{F}_i(U) \to \coprod \mathcal{G}_i(U)$ exact (AB 4 in Ab!)

$\Longrightarrow 0 \to \coprod \mathcal{F}_i^{pre} \to \coprod \mathcal{G}_i^{pre}$ exact (Presheaf coproduct is pointwise)

$\Longrightarrow 0 \to \coprod \mathcal{F}_i \to \coprod \mathcal{G}_i$ exact (Sheafification exact)

$\boxed{\text{Note}}$ (a)* Step 1 fails for direct product analog.

(b)* Rest are similar.

(c)* Presheaf limits of sheaves are sheaves as sheaf condition is a limit diagram & lims commute!

See much of theorem is formal.
But some is special. Many examples where

$$(\text{Ab}6),\ (\text{Ab}4^*)\ \&\ (\text{Compact Proj Gens})$$

fail. In fact, the latter fails often:

<u>Example</u>

- Let X be locally connected & $x \in X$ a closed point with no minimal open neighborhood. (E.g., x a manifold point).
- We will observe there is no surjection $P \longrightarrow \mathbb{Z}_X$ where P is a projective sheaf & $\mathbb{Z}_X$ constant sheaf on X.

For any $x \in U \subseteq X$ choose $x \in V \subsetneq U$ smaller nhood.

Consider

$$j : V \hookrightarrow X$$
$$i : X \setminus \{x\} \longrightarrow X.$$

As ~~the~~ V & $X \setminus \{x\}$ cover X we have a surjection

$$\begin{tikzcd} & j_! \mathbb{Z}_V \oplus i_! \mathbb{Z}_{X \setminus \{x\}} \arrow[d] \\ P \arrow[ur, dashed, "?"] \arrow[r] & \mathbb{Z}_X \end{tikzcd}$$

check on stalks!

So map from P lifts (by projectivity).
Evaluate on U. Since $U \not\subseteq V$ & $U \not\subseteq X \setminus \{x\}$ have

$$j_! \mathbb{Z}_V(U) = i_! \mathbb{Z}_{X \setminus \{x\}}(U) = 0 \qquad \text{so} \qquad P(U) \longrightarrow \mathbb{Z}_X(U)$$

must be 0 map. Taking colimit over such U shows

$$P_x \xrightarrow{\ 0\ } \mathbb{Z}_{X,x} = \mathbb{Z}$$

is 0 map, so not surjective.

$\boxed{\text{Remark}}$ $\mathcal{A}$ an abelian category. $\mathcal{A}$ has a single compact projective generator $\iff \mathcal{A} \simeq \mathrm{Mod}_R$

$$\begin{tikzcd}
\left(\mathcal{A} \text{ w/ generator } G\right) \arrow[rr, bend right, "R = \mathrm{End}(G)"'] & & \left(\mathrm{Mod}_R\right) \arrow[ll, bend right, "G = R"']
\end{tikzcd}$$

So in a sense, Cond(Ab) closer to a Module category than a sheaf category (subjective!!)

<u>The Condensed Picture</u>

$\boxed{\text{Recall}}$

$$\mathrm{Cond}(\mathrm{Ab}) \simeq \mathrm{Ab}(\mathrm{EDSet})$$

Let $\mathcal{F} \in \mathrm{PreAb}(\mathrm{EDSet})$. Then $\mathcal{F}$ is a sheaf

$\iff \forall\ U, V \in \mathrm{EDSet}$

$$\mathcal{F}(U \amalg V) \longrightarrow \mathcal{F}(U) \times \mathcal{F}(V)$$

is an iso

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i.e., the entire sheaf condition boils down to this very simple type of cover.

(use: surjections of EDSets split).

This makes Ab(EDset) behave in a presheafy way!

Lemma

Limits and Colimits in $Ab(EDSet)$ are computed pointwise.

Proof

Already true for limits in any sheaf category.
(See note (c) one page 4)

Let $\mathcal{F}_i$ be a diagram of sheaves. It suffices to show that the presheaf

$$U \longmapsto \operatorname{colim} \mathcal{F}_i(U)$$

is a sheaf (since the sheaf colimit is the sheafification of it, see example (AB3) on page 4).
But the sheaf condition is easy to check:

Let $U, V \in EDSet$:

$$
\begin{aligned}
\operatorname{Colim}(\mathcal{F}_i(U \amalg V)) &= \operatorname{colim}(\mathcal{F}_i(U) \times \mathcal{F}_i(V)) \\
&= \operatorname{Colim}(\mathcal{F}_i(U) \amalg \mathcal{F}_i(V)) \\
&= \operatorname{colim}\mathcal{F}_i(U) \amalg \operatorname{colim}\mathcal{F}_i(V) \\
&= \operatorname{Colim}\mathcal{F}_i(U) \times \operatorname{Colim}\mathcal{F}_i(V)
\end{aligned}
$$

finite prods & coprods agree in Ab

colims & $\amalg$ always commute

So we win! $\varnothing$

Corollary $\mathcal{F} \xrightarrow{\pi} \mathcal{G}$ surjects $\iff$ $\mathcal{F}(S) \xrightarrow{\pi(S)} \mathcal{G}(S)$ does for every $S \in EDSet$

Pf/ π surjects $\iff$ $\operatorname{coker}\pi = 0$ $\overset{\text{coker is a colim}}{\iff}$ $\operatorname{coker}\pi(U) = 0$ $\forall U$ $\iff$ $\pi(U)$ surjects $\forall U$

Proof of Main Theorem

Since lims/colims/exactness/... computed pointwise ~~in~~ and $Ab(EDset)$, everything except (Compact Projective Generators) follows formally from ~~[illegible]~~ case of Ab.

(Cf. Remark 2 on Page 3)

For example

$\mathcal{F}_i \longrightarrow \mathcal{G}_i \longrightarrow 0$ diagram of surjections in ~~Ab~~ $Cond(Ab)$

$\Longrightarrow \mathcal{F}_i(U) \longrightarrow \mathcal{G}_i(U) \longrightarrow 0$ diagram of surj in Ab
$\forall U$ (by Corollary on p. 2)

$\Longrightarrow \prod \mathcal{F}_i(U) \longrightarrow \prod \mathcal{G}_i(U) \longrightarrow 0$ surj in Ab
$\forall U$ (since Ab is $(AB4^*)$)

$\Longrightarrow \prod \mathcal{F}_i \longrightarrow \prod \mathcal{G}_i \longrightarrow 0$ (by Lemma on p. 7).

So $Cond(Ab)$ is $(AB4^*)$

Compact Projective Objects

$\boxed{\text{Lemma A}}$

The forgetful functor

$$Cond(Ab) \longrightarrow Cond(Set)$$

has a left adjoint $S \longmapsto \mathbb{Z}[S] : (U \longmapsto \mathbb{Z}[S(U)])^{\#}$

Proof This follows from adjoint functor theorem as Forget commutes w/ limits. One can check directly that the given rule satisfies adjointness, ~~Indeed, [illegible]~~ free group sheafification.

Indeed, letting $\mathcal{C} = \mathrm{CHaus}$, PFSet or EDSet, have:

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Ab}(\mathcal{C})}(\mathbb{Z}[S], A) &= \mathrm{Hom}_{\mathrm{PreAb}(\mathcal{C})}(\mathbb{Z}[S]^{pre}, A) \\
&= \{\mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}[S(U)], A(U)) \mid U \in \mathcal{C}\} \\
&= \{\mathrm{Hom}_{\mathrm{Set}}(S(U), A(U)) \mid U \in \mathcal{C}\} \\
&= \mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(S, A)
\end{aligned}$$

compatible

$\boxed{\text{Lemma B}}$ $M \in \mathrm{CHaus}$. Then $\forall\ A \in \mathrm{Cond}(\mathrm{Ab})$

$$\mathrm{Hom}_{\mathrm{Cond}(\mathrm{Ab})}(\mathbb{Z}[M], A) = \mathrm{Hom}_{\mathrm{Cond}(\mathrm{Set})}(M, A) = A(M)$$

$\boxed{\text{Lemma C}}$ $M \in \mathrm{EDSet}$. Then $\mathbb{Z}[M]$ is compact and projective.

Pf/ By Lemma B, the functor

$$\mathrm{Hom}(M, -) : A \longmapsto A(M)$$

This commutes with all limits and colimits by Lemma on page 4, giving the proof.

It remains to show these generate $\mathrm{Cond}(\mathrm{Ab})$.

$\boxed{\text{Recall}}$ A set $\{M_i\}_I$ generate $\mathcal{A}$ if $\forall\ A \in \mathcal{A}$, $\exists$ a surjection $\bigoplus M_i \twoheadrightarrow A$.

Proof of "Enough"-ness

Let $T \in \mathrm{Cond}(\mathrm{Ab})$.

By Zorn, pick maximal $T' \subseteq T$ admitting a surjection $\bigoplus \mathbb{Z}[M_i] \twoheadrightarrow T'$ w/ $M_i \in \mathrm{EDSet}$.

If $T/T' \neq 0$, this must be witnessed by evaluating on some $N \in \mathrm{EDSet}$.

So $0 \neq (T/T')(N) = \mathrm{Hom}(\mathbb{Z}[N], T/T')$

$$\underset{\phi \neq 0.}{\rotatebox{90}{$\in$}}$$

$$\begin{tikzcd} & T \arrow[d] \\ \mathbb{Z}[N] \arrow[r, "\phi"'] \arrow[ur, dotted, "\exists \tilde{\Phi}"] & T/T' \end{tikzcd}$$

$\tilde{\Phi}$ $\exists$ by projectivity.

Since $\phi \neq 0$, $\tilde{\Phi} \nsubseteq T'$, so if $T'' = T' + \mathrm{im}\,\tilde{\Phi}$

have $T' \subsetneq T'' \subseteq T$

& $\left(\bigoplus \mathbb{Z}[M_i]\right) \oplus \mathbb{Z}[N] \twoheadrightarrow T''$

contradicting maximality of T' ↯

Cohomology

Now have a nice setting to do homological algebra.

We'd like to define, for $X \in \mathrm{CHaus}$ ~~(or CGTop even?)~~

$$M \in \mathrm{Cond}(\mathrm{Ab})$$

$$H^i_{cond}(X, M)$$

$\boxed{\text{Rmk}}$ Some question about enough injectives.
Perhaps set theoretic difficulties?

That said, if we want to derive $\Gamma_{cond}(X, M)$,
we can observe by $\boxed{\text{Lemma B}}$

$$\Gamma_{cond}(X, M) = M(X) = \mathrm{Hom}_{\mathrm{Cond}(\mathrm{Ab})}(\mathbb{Z}[X], M)$$

We have enough projectives, & so can resolve $\mathbb{Z}[X]$.
Thus:

$$H^i_{cond}(X, M) = R^i\Gamma_{cond}(X, M)$$

$$= R^i \mathrm{Hom}_{\mathrm{Cond}(\mathrm{Ab})}(\mathbb{Z}[X], M)$$

$$= \mathrm{Ext}^i_{\mathrm{Cond}(\mathrm{Ab})}(\mathbb{Z}[X] M).$$

In fact, we can do this quite explicitly!
Recall, any $X \in \mathrm{CHaus}$ admits a surjection from an EDSet:
$\beta X^{disc} \twoheadrightarrow X$, w/ βX^{disc} the Stone-Cech compactification of X^{disc}

Then we can somewhat canonically form the resolution

$$\cdots \quad S_2 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow \\ \longleftarrow \\ \longleftarrow}\; \underset{\beta(S_0 \times_X S_0)^{disc}}{\underset{\parallel}{S_1}} \;\substack{\longrightarrow\\ \longrightarrow \\ \longleftarrow}\; \underset{\beta X^{dis}}{\underset{\parallel}{S_0}} \longrightarrow X$$

Giving

$$\cdots \longrightarrow \mathbb{Z}[S_2] \longrightarrow \mathbb{Z}[S_1] \longrightarrow \mathbb{Z}[S_0] \longrightarrow \mathbb{Z}[X]$$

a projective resolution! Then

$$H^i_{cond}(X, M) = Ext^i_{cond(Ab)}(\mathbb{Z}[X], M)$$

$$= H^i(Hom(\mathbb{Z}[S_0], M) \longrightarrow Hom(\mathbb{Z}[S_1], \cancel{Ab}) \longrightarrow \cdots)$$

$$= H^i(M(S_0) \longrightarrow M(S_1) \longrightarrow M(S_2) \longrightarrow \cdots)$$