

# Condensed Abelian Groups

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- Last time saw condensed sets gave a nice categorical framework to do topology.
- This time see condensed abelian groups give a nice categorical framework to do topological algebra.

## Example

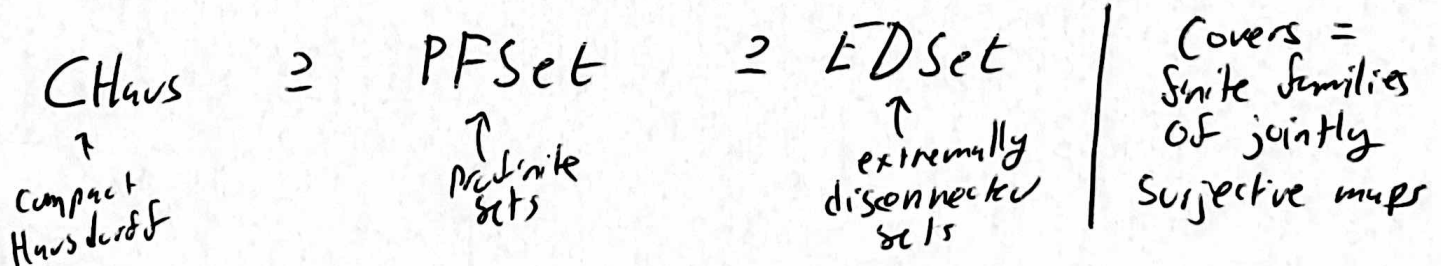
Saw a glimpse of this last week.

$$0 \longrightarrow \mathbb{R}^{\text{disc}} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{R}^{\text{disc}} \longrightarrow 0$$

- 1) A location where  $\mathbb{R}/\mathbb{R}^{\text{disc}}$  exists
- 2) As these are sheaves, evaluate @ points/opens to probe structure of  $\mathbb{R}/\mathbb{R}^{\text{disc}}$  in terms of  $H'(-, \mathbb{R}^{\text{disc}}) \rightarrow H'(-, \mathbb{R})$

In particular, we benefit from having a robust setting in which to do homological algebra.

## Recall Defined sites



$$\text{Cond}(Ab) = Ab(\text{CHaus}) = Ab(\text{PFSet}) = Ab(\text{EDSet}).$$

# Main Theorem for Today

Cond(Ab) is an abelian category satisfying

(Ab3) Colimits exist

(Ab4) Coproducts are exact

(Ab5) Filtered Colimits are exact

(Ab6)  $\mathcal{J}$  index,  $I_j$  family of filtered categories:

$$\operatorname{colim}_{i \in \prod_j I_j} \prod_{j \in \mathcal{J}} M_{i,j} \xrightarrow{\cong} \prod_{j \in \mathcal{J}} \operatorname{colim}_{i \in I_j} M_{i,j} \text{ iso}$$

(Ab3\*) Limits exist

(Ab4\*) Products are exact.

(Symmetric Monoidal)  $\otimes$  exists w/ relevant diagrams and unit.

(Closed) Internal Hom exists.

(Compact Projective Generators)  $\overset{\text{in particular}}{\implies}$  Enough Projectives

**Def<sup>n</sup>**  $\mathcal{A}$  an abelian category.

$M \in \mathcal{A}$  is

1) Compact if ~~where~~  $\operatorname{Hom}(M, -)$  commutes w/ filtered colimits

2) Projective if  $\operatorname{Hom}(M, -)$  is exact.

# Remarks

(3)

- ① The category of abelian groups (more generally  $R$ -modules) satisfies the theorem.
- ② Let  $\mathcal{C}$  be a category, and  $\text{PreAb}(\mathcal{C})$  the category of presheaves of abelian groups. Then  $\text{PreAb}(\mathcal{C})$  satisfies the theorem.

? Limits/Colimits/Injectivity/Surjectivity ... etc computed pointwise, so result follows formally from the case for  $\text{Ab}$ .

- ③ Let  $\mathcal{C}$  be a (small) site. Then much of the theorem holds formally for  $\text{Ab}(\mathcal{C})$ , but not all! In particular,  $\text{Ab}(\mathcal{C})$  is abelian & satisfies

(AB3), (AB4), (AB5), (AB3\*) (Symmetric Monoidal) (Closed).

But not in general.

(AB4\*), (AB6), (Compact Projective Generators)

It is these last three that make  $\text{Cond}(\text{Ab})$  special!

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Before moving into the condensed setting, discuss the formality of much of the theorem for  $\text{Ab}(\mathcal{C})$ . It follows essentially from the ~~existence~~ result on ~~PreAb~~  $\text{PreAb}(\mathcal{C})$  together w/ the exactness & adjointness properties of the sheafification functor  $\#$ .

**Example** (AB3) for  $Ab(\mathcal{C})$ .

let  $\mathcal{F}_i$  be a system of sheaves. We  
Claim  $(\text{colim}_i \mathcal{F}_i^{\text{pre}})^{\#}$  satisfies colimit property  
          ↑                                  ↑  
          presheaf                          sheafification.  
          colimit

Indeed

$$\text{Hom}_{Ab(\mathcal{C})}((\text{colim}_i \mathcal{F}_i^{\text{pre}})^{\#}, \mathcal{G}) = \text{Hom}_{\text{Pre}Ab(\mathcal{C})}(\text{colim}_i \mathcal{F}_i^{\text{pre}}, \mathcal{G}^{\text{pre}}) \quad (\text{Adjoints})$$

$$= \lim_i \text{Hom}_{\text{Pre}Ab(\mathcal{C})}(\mathcal{F}_i^{\text{pre}}, \mathcal{G}^{\text{pre}})$$

$$= \lim \text{Hom}_{Ab(\mathcal{C})}(\mathcal{F}_i, \mathcal{G})$$

- (AB4)  $0 \rightarrow \mathcal{F}_i \rightarrow \mathcal{G}_i$  exact
- $\Rightarrow 0 \rightarrow \mathcal{F}_i(U) \rightarrow \mathcal{G}_i(U)$  exact ( $\Gamma$  left exact)
  - $\Rightarrow 0 \rightarrow \coprod \mathcal{F}_i(U) \rightarrow \coprod \mathcal{G}_i(U)$  exact (AB4 in  $Ab!$ )
  - $\Rightarrow 0 \rightarrow \coprod \mathcal{F}_i^{\text{pre}} \rightarrow \coprod \mathcal{G}_i^{\text{pre}}$  exact (presheaf coproduct is pointwise)
  - $\Rightarrow 0 \rightarrow \coprod \mathcal{F}_i \rightarrow \coprod \mathcal{G}_i$  exact (Sheafification exact)

**Note** (a)\* Step 1 fails for direct product analog.

(b)\* Rest are similar.

(c)\* Presheaf limits of sheaves are sheaves as sheaf condition is a limit diagram & lms commute!

See much of theorem is formal.

But some is special. Many examples where

(Ab6), (Ab4\*) & (Compact Proj Gens)

fail. In fact, the latter fails often:

Example

• Let  $X$  be locally connected &  $x \in X$  a closed point with no minimal open neighborhood. (E.g.,  $x$  a manifold point).

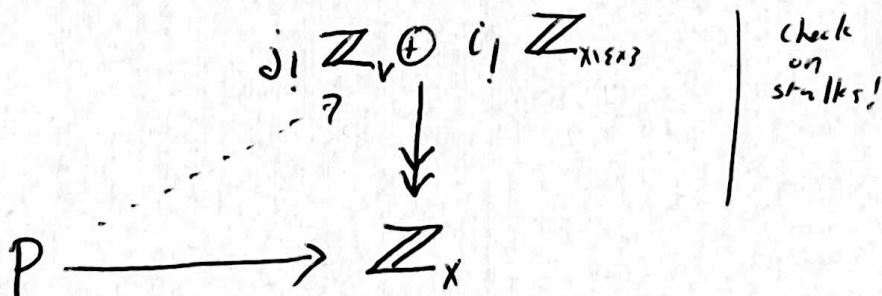
• We will observe there is no surjection  $P \rightarrow \mathbb{Z}_x$  where  $P$  is a projective sheaf &  $\mathbb{Z}_x$  constant sheaf on  $X$ .

For any  $x \in U \subseteq X$  choose  $x \in V \subseteq U$  smaller nhood.

Consider

$$j: V \hookrightarrow X$$
$$i: X \setminus \{x\} \rightarrow X$$

As  $U \subseteq V$  &  $X \setminus \{x\}$  cover  $X$  we have a surjection



So map from  $P$  lifts (by projectivity).

Evaluate on  $U$ . Since  $U \not\subseteq V$  &  $U \not\subseteq X \setminus \{x\}$  have

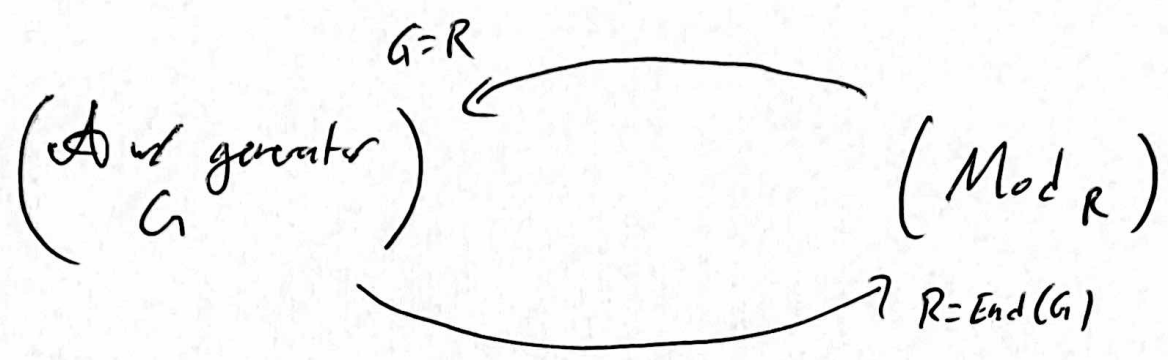
$$j! \mathbb{Z}_V(U) = i! \mathbb{Z}_{X \setminus \{x\}}(U) = 0 \quad \text{so } P(U) \rightarrow \mathbb{Z}_x(U)$$

must be 0 map. Taking colimit over such  $U$

shows

$$P_x \xrightarrow{0} \mathbb{Z}_{X \setminus \{x\}} = \mathbb{Z} \quad \text{is 0 map, so not surjective.}$$

**Remark**  $\mathcal{A}$  an abelian category.  $\mathcal{A}$  has a single compact projective generator  $\iff \mathcal{A} \simeq \text{Mod } R$



So in a sense,  $\text{Cond}(\text{Ab})$  closer to a Module category than a sheaf category (subjective!!)

The Condensed Picture

**Recall**

$$\text{Cond}(\text{Ab}) \simeq \text{Ab}(\text{EDSet})$$

Let  $\mathcal{F} \in \text{PreAb}(\text{EDSet})$ . Then  $\mathcal{F}$  is a sheaf

$$\iff \forall U, V \in \text{EDSet}$$

$$\mathcal{F}(U \amalg V) \longrightarrow \mathcal{F}(U) \times \mathcal{F}(V)$$

is an iso

i.e., the entire sheaf condition boils down to this very simple type of cover.

(use: surjections of EDsets split).

This makes  $\text{Ab}(\text{EDSet})$  behave in a perfectly way!

**Lemma**

Limits and colimits in  $Ab(EDSet)$  are computed pointwise.

Proof

Already true for limits in any sheaf category.  
(see note (c) one page (4))

Let  $\mathcal{F}_i$  be a diagram of sheaves. It suffices to show that the presheaf

$$U \longmapsto \text{colim } \mathcal{F}_i(U)$$

is a sheaf (since the sheaf colimit is the sheafification of it, see example (A3) on page (4)).

But the sheaf condition is easy to check:

Let  $U, V \in EDSet$ :

$$\begin{aligned}
 \text{colim}(\mathcal{F}_i(U \amalg V)) &= \text{colim}(\mathcal{F}_i(U) \times \mathcal{F}_i(V)) \\
 &= \text{colim}(\mathcal{F}_i(U) \amalg \mathcal{F}_i(V)) \quad \leftarrow \begin{array}{l} \text{finite presheaves} \\ \& \text{coproducts} \\ \text{agree in } Ab \end{array} \\
 &= \text{colim } \mathcal{F}_i(U) \amalg \text{colim } \mathcal{F}_i(V) \quad \leftarrow \begin{array}{l} \text{colims } \& \amalg \\ \text{always commute} \end{array} \\
 &= \text{colim } \mathcal{F}_i(U) \times \text{colim } \mathcal{F}_i(V)
 \end{aligned}$$

So we win!  $\square$

**Corollary**

$$\mathcal{F} \xrightarrow{\pi} \mathcal{G} \text{ surjects } \iff \mathcal{F}(S) \xrightarrow{\pi(S)} \mathcal{G}(S) \text{ does for every } S \in EDSet$$

$$\text{PS } \pi \text{ surjects } \iff \text{coker } \pi = 0 \iff \begin{array}{l} \text{coker is a colim} \\ \text{coker } \pi(U) = 0 \iff \pi(U) \text{ surjects } \forall U \end{array}$$

# Proof of Main Theorem

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Since limits/colimits/exactness/... computed pointwise ~~in~~  $Ab$  and  $Ab(EDset)$ , everything except (Compact Projective Generators) follows formally from ~~over~~ case of  $Ab$ .  
(Cf. Remark ② on page ③)

For example

$\mathcal{F}_i \rightarrow \mathcal{G}_i \rightarrow 0$  diagram of surjections in  ~~$Ab$~~   $Cond(Ab)$

$\Rightarrow \mathcal{F}_i(u) \rightarrow \mathcal{G}_i(u) \rightarrow 0$  diagram of surj in  $Ab$   
 $\forall u$  (by Corollary on p. ②)

$\Rightarrow \pi \mathcal{F}_i(u) \rightarrow \pi \mathcal{G}_i(u) \rightarrow 0$  surj in  $Ab$   
 $\forall u$  (since  $Ab$  is  $(AB)^*$ )

$\Rightarrow \pi \mathcal{F}_i \rightarrow \pi \mathcal{G}_i \rightarrow 0$  (by Lemma on p. ④)

So  $Cond(Ab)$  is  $(AB)^*$

## Compact Projective Objects

### Lemma A

The Forgetful Functor

$$Cond(Ab) \longrightarrow Cond(Set)$$

has a left adjoint  $S \longmapsto \mathbb{Z}[S] : (U \longmapsto \mathbb{Z}[S(u)])^\#$

Proof This follows from adjoint functor theorem as

Forget commutes w/ limits. One can check directly that the given rule satisfies adjointness. ~~... see group structure~~



Indeed, letting  $\mathcal{C} = (\text{Haus}, \text{PFset}$  or  $\text{EDset}$ , have:

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$$\text{Hom}_{\text{Ab}(\mathcal{C})}(\mathbb{Z}[S], A) = \text{Hom}_{\text{PreAb}(\mathcal{C})}(\mathbb{Z}[S]^{\text{pre}}, A)$$

$$= \left\{ \text{Hom}_{\text{Ab}}(\mathbb{Z}[S(u)], A(u)) \mid u \in \mathcal{C} \right\} \leftarrow \text{complete}$$

$$= \left\{ \text{Hom}_{\text{Set}}(S(u), A(u)) \mid u \in \mathcal{C} \right\} \leftarrow$$

$$= \text{Hom}_{\text{sh}(\mathcal{C})}(S, A) \quad \text{QED}$$

**Lemma B**  $M \in \text{Haus}$ . Then  $\forall A \in \text{Cond}(\text{Ab})$

$$\text{Hom}_{\text{Cond}(\text{Ab})}(\mathbb{Z}[M], A) = \text{Hom}_{\text{Cond}(\text{Set})}(M, A) = A(M)$$

**Lemma C**  $M \in \text{EDSet}$ . Then  $\mathbb{Z}[M]$  is compact and projective.

*Pf* By Lemma B, the functor

$$\text{Hom}(M, -) : A \longrightarrow A(M)$$

This commutes with all limits and colimits by Lemma on page (4), giving the proof.

It remains to show these generate  $\text{Cond}(\text{Ab})$ .

**Recall** A set  $\{M_i\}_I$  generate  $\mathcal{A}$  if  $\forall A \in \mathcal{A}$ ,

$$\exists \text{ a surjection } \bigoplus M_i \longrightarrow A.$$

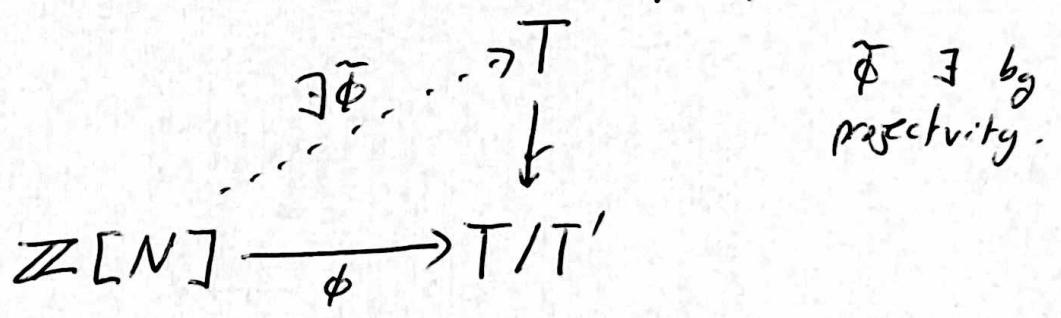
Proof of "Enoughness"

Let  $T \in \text{Cond}(Ab)$ .

By Zorn, pick maximal  $T' \subseteq T$  admitting a surjection  $\bigoplus \mathbb{Z}[M_i] \twoheadrightarrow T'$  w/  $M_i \in \text{EDSet}$ .

If  $T/T' \neq 0$ , this must be witnessed by evaluating on some  $N \in \text{EDSet}$ .

So  $0 \neq (T/T')(N) = \text{Hom}(\mathbb{Z}[N], T/T')$   
 $\downarrow$   
 $\phi \neq 0$ .



Since  $\phi \neq 0$ ,  $\tilde{\phi} \not\subseteq T'$ , so if  $T'' = T' + \text{im } \tilde{\phi}$

have  $T' \subsetneq T'' \subseteq T$

&  $(\bigoplus \mathbb{Z}[M_i]) \oplus \mathbb{Z}[N] \twoheadrightarrow T''$

contradicting maximality of  $T'$   $\Downarrow$



# Cohomology

(11)

Now have a nice setting to do homological algebra.

We'd like to define, for  $X \in \text{CHaus}$  ~~(or  $\text{CTop}$ ?)~~  
 $M \in \text{Cond}(\text{Ab})$

$$H_{\text{cond}}^i(X, M)$$

Rmk Some question about enough injectives.

Perhaps set theoretic difficulties?

That said, if we want to derive  $\Gamma_{\text{cond}}^i(X, M)$ ,  
we can observe by Lemma B

$$\Gamma_{\text{cond}}^i(X, M) = M(X) = \text{Hom}_{\text{cond}(\text{Ab})}(\mathbb{Z}[X], M)$$

We have enough projectives, & so can resolve  $\mathbb{Z}[X]$ .  
Thus:

$$\begin{aligned} H_{\text{cond}}^i(X, M) &= R^i \Gamma_{\text{cond}}^i(X, M) \\ &= R^i \text{Hom}_{\text{cond}(\text{Ab})}(\mathbb{Z}[X], M) \\ &= \text{Ext}_{\text{cond}(\text{Ab})}^i(\mathbb{Z}[X], M) \end{aligned}$$

In fact, we can do this quite explicitly!

Recall, any  $X \in \text{CHaus}$  admits a surjection from an ED&T:  
 $BX^{\text{disc}} \rightarrow X$  w/  $BX^{\text{disc}}$  the Stone-Čech compactification of  $X^{\text{disc}}$

Then we can somewhat canonically form  
the resolution

$$\begin{array}{ccccccc}
 S_2 & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & S_1 & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & S_0 & \longrightarrow & X \\
 & & \begin{array}{c} \curvearrowright \\ \parallel \\ \curvearrowleft \end{array} & & \begin{array}{c} \parallel \\ \beta X^{dis} \end{array} & & \\
 & & \beta(S_0 \times S_0)^{disc} & & & & 
 \end{array}$$

Giving

$$\dots \longrightarrow \mathbb{Z}[S_2] \longrightarrow \mathbb{Z}[S_1] \longrightarrow \mathbb{Z}[S_0] \longrightarrow \mathbb{Z}[X]$$

a projective resolution! Then

$$H_{cond}^i(X, M) = \text{Ext}_{\text{cond}(Ab)}^i(\mathbb{Z}[X], M)$$

$$= H^i(\text{Hom}(\mathbb{Z}[S_0], M) \longrightarrow \text{Hom}(\mathbb{Z}[S_1], M) \longrightarrow \dots)$$

$$= H^i(M(S_0) \longrightarrow M(S_1) \longrightarrow M(S_2) \longrightarrow \dots)$$