Condensed Talk 9/9/21

1. Recall:

- $*_{\text{proét}} = \text{Pro}(\text{Fin}) \subseteq \text{Top}$ full subcategory of compact Hausdorff totally disconnected spaces e.g. $\hat{\mathbb{Z}}$.

- A cover is a finite family $\{U_i \to U\}_{i=1}^{n}$ of maps of profinite sets such that
$$\coprod_{i=1}^{n} U_i \longrightarrow U$$
is surjective.

- A condensed set is a sheaf on $*_{\text{proét}}$, i.e.
$$F : *_{\text{proét}} \to \text{Sets}^{op}$$
such that $\forall$ cover $\{U_i \to U\}_{i=1}^{n}$, the map
$$F(U) \to eq\left(\prod_{i=1}^{n} F(U_i) \rightrightarrows \prod_{i,j=1}^{n} F(U_i \times_U U_j)\right)$$
is an isomorphism.

- $\text{Cond}(\text{Sets}) \subseteq \text{Fun}(*_{\text{proét}}, \text{Sets}^{op})$
i.e. a morphism of condensed sets is just a natural transformation.

Today: Relationship between condensed sets and topological spaces.

2. Spaces as condensed sets

$$\begin{tikzcd} \mathrm{Top} \arrow[r, hook] & \mathrm{Fun}(\mathrm{Top}, \mathrm{Sets}^{op}) \arrow[r, "\mathrm{Res}"] & \mathrm{Fun}(*_{\text{proét}}, \mathrm{Sets}^{op}) \\ X & \longmapsto & \underline{X} = (S \mapsto \mathrm{Hom}_{\mathrm{Top}}(S, X)) \end{tikzcd}$$

Prop 2.1 If X is a space, $\underline{X}$ is a condensed set.

Proof: We must check the sheaf condition for $\underline{X}$.

(i) Let $U = \coprod_{i=1}^{n} U_i$, $U_i \in *_{\text{proét}}$.

$$\begin{aligned} \underline{X}(U) = \mathrm{Hom}_{\mathrm{Top}}(U, X) &= \mathrm{Hom}_{\mathrm{Top}}\left(\coprod_{i=1}^{n} U_i, X\right) \\ &\cong \prod_{i=1}^{n} \mathrm{Hom}_{\mathrm{Top}}(U_i, X) \\ &= \prod_{i=1}^{n} \underline{X}(U_i). \quad \checkmark \end{aligned}$$

(ii) Let $U' \xrightarrow{\pi} U$ be a surjection of profinite sets. We must check that

$$\mathrm{Hom}_{\mathrm{Top}}(U, X) \longrightarrow \mathrm{Hom}_{\mathrm{Top}}(U', X) \rightrightarrows \mathrm{Hom}_{\mathrm{Top}}(U' \times_{U} U', X)$$

is an equalizer diagram. That is, we must show that a map

$$f : U' \longrightarrow X$$

factors through π iff

$$(*) \qquad p_1 f = p_2 f \qquad\qquad p_i : U' \times_{U} U' \to U'.$$

$$U' \times_{U} U' = \{(u, v) \in U' \times U' \mid \pi(u) = \pi(v)\}$$

So, (*) is equivalent to

$$\pi(u) = \pi(v) \implies f(u) = f(v).$$

Thus, (*) is equivalent to f factoring through a map

$$\bar{f} : U \longrightarrow X \qquad \bar{f}(u) = f(\tilde{u}) \text{ for } \pi(\tilde{u}) = u.$$

set theoretically. Therefore, we must show that a map

$$\bar{f} : U \to X$$

is continuous as soon as $U' \overset{\pi}{\twoheadrightarrow} U \to X$ is continuous.

General topology: A surjection of compact Hausdorff spaces is a quotient map □

So, we have constructed a functor

$$\begin{aligned} \Gamma : \mathrm{Top} &\longrightarrow \mathrm{Cond(Sets)} \\ X &\longmapsto \underline{X} \end{aligned}$$

Prop 2.2 Γ has a left adjoint Θ. The counit

$$\Theta\Gamma(X) \longrightarrow X$$

Before proving 2.2, we must recall some general topology.

Def: A space X is *compactly generated* if for any map $f : X \to Y$, TFAE

(i) f is continuous

(ii) $\forall$ compact Hausdorff $S \longrightarrow X$, the composition $S \to X \to Y$ is continuous.

$\mathrm{CGTop} \subseteq \mathrm{Top}$:= full subcategory of compactly generated spaces.

<u>Fact</u>: The inclusion CGTop ⊆ Top has a left adjoint:

$$\begin{aligned} \mathrm{Top} &\longrightarrow \mathrm{CGTop} \\ X &\longmapsto X^{cg} \end{aligned}$$

Here, $|X^{cg}| = |X|$, but we give X^{cg} the quotient topology from the map

$$\coprod_{\substack{S \to X \\ S \text{ compact Hausdorff}}} S \twoheadrightarrow X^{cg}$$

<u>Rmk</u>: • Since every compact Hausdorff space admits a continuous surjection (hence quotient map) from a profinite set (its Stone-Čech compactification), we may replace the condition "compact Hausdorff" in the def. of "compactly generated" and of the functor $(-)^{cg}$ by "profinite".

• If X is CG, the counit $X^{cg} \to X$ is an isomorphism.

<u>Proof of 2.2</u>: Let $S \in \mathrm{Cond}(\mathrm{Set})$.

$|\Theta(S)| = S(*)$.

We give $S(*)$ the quotient topology from the map

$$\coprod_{\substack{\underline{T} \to S \\ T \text{ profinite set}}} T \longrightarrow S(*) \qquad (\text{note } \underline{T}(*) = T \text{ for } T \text{ profinite})$$

Let X be a space.

$$\mathrm{Hom}_{\mathrm{Top}}(\Theta(S), X) = \left\{ f : S(*) \to X \;\middle|\; \begin{array}{l} \forall \text{ profinite } T, \\ \underline{T} \to S \end{array} \;\; \begin{array}{l} T = \underline{T}(*) \to S(*) \xrightarrow{f} X \\ \text{is cont.} \end{array} \right\}$$

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Cond}}(S, \underline{X}) &\longrightarrow \mathrm{Hom}_{\mathrm{Top}}(\Theta(S), X) \\ F : S \to \underline{X} &\longmapsto F(*) : S(*) \longrightarrow \underline{X}(*) = X \end{aligned}$$

If T is profinite, $\underline{T} \to S$, the composite

$$\underline{T} \xrightarrow{\eta} S \longrightarrow \underline{X} \qquad \in \mathrm{Hom}_{\mathrm{Cond(Sets)}}(\underline{T}, \underline{X}) = \mathrm{Hom}_{\mathrm{Top}}(T, X), \qquad = F(\eta).$$

evaluates on $*$ to give

$$\begin{tikzcd} T \arrow[r] \arrow[rr, bend right, "F(\eta)"'] & S(*) \arrow[r] & X \end{tikzcd}$$

which is continuous. For the inverse

$$\mathrm{Hom}_{\mathrm{Top}}(\Theta(S), X) \longrightarrow \mathrm{Hom}_{\mathrm{Cond}}(S, \underline{X})$$

suppose given $f : \Theta(S) \to X$. Define

$$\rho_f : S \longrightarrow \underline{X}$$

by

$$\begin{tikzcd} \rho_f(T) : S(T) \arrow[r] \arrow[d, phantom, "\parallel"] & \underline{X}(T) \arrow[d, phantom, "\parallel"] \\ \mathrm{Hom}_{\mathrm{Cond}}(\underline{T}, S) \arrow[r] & \mathrm{Hom}_{\mathrm{Top}}(T, X) \\ \underline{T} \to S \arrow[r, mapsto] & T \to S(*) \xrightarrow{f} X \end{tikzcd}$$

(cont. by ass.) $\square$

Prop 2.3: (a) Γ is faithful. (b) $\Gamma|_{CGTop}$ is fully faithful.

Proof (a) Let X, Y be spaces.

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{Top}}(X, Y) \arrow[r, "\Gamma"] \arrow[rrd, bend right, "\mathrm{inj}"] & \mathrm{Hom}_{\mathrm{Cond(Sets)}}(\underline{X}, \underline{Y}) \arrow[r, "ev_*"] & \mathrm{Hom}_{\mathrm{Sets}}(\underline{X}(*), \underline{Y}(*)) \arrow[d, phantom, "\parallel"] \\ & & \mathrm{Hom}_{\mathrm{Sets}}(X, Y) \end{tikzcd}$$

ⓑ If $X \in \mathrm{Top}$,

$$\Theta\Gamma(X) = \underline{X}(*) \quad \text{w/ the quotient topology from } \bigsqcup_{\substack{\underline{S}\to\underline{X} \\ S \text{ profinite}}} S \to \underset{\substack{\shortparallel \\ X}}{\underline{X}(*)}$$

i.e. X w/ the quotient topology from $\displaystyle\bigsqcup_{\substack{S\to X \\ \text{profinite}}} S \to X$

i.e. X^{cg}

Furthermore, the counit $X^{cg} \to X$ is the one from the adjunction $\mathrm{CGTop} \rightleftarrows \mathrm{Top}$.

In particular, if $X \in \mathrm{CGTop}$, $\Theta\Gamma(X) \xrightarrow{\sim} X$.

So for $X, Y \in \mathrm{CGTop}$,

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Cond(Sets)}}(\Gamma(X), \Gamma(Y)) &\simeq \mathrm{Hom}_{\mathrm{Top}}(\Theta\Gamma(X), Y) \\ &\cong \mathrm{Hom}_{\mathrm{Top}}(X, Y) \quad \square \end{aligned}$$

Diagram:

$$\begin{tikzcd}
\mathrm{Top} \arrow[r, bend right, "\Gamma"'] \arrow[r, phantom, "\perp"] & \mathrm{Cond(Sets)} \arrow[l, bend right, "\Theta"'] \\
\mathrm{CGTop} \arrow[u, hook] \arrow[ur, hook]
\end{tikzcd}$$

3. QCQS condensed sets

Def (a) A condensed set T is quasi-compact if $\exists$ a surjection

$$\underline{S} \twoheadrightarrow T$$

for S a profinite set.

(b) A condensed set T is quasi-separated if $\forall$ profinite sets

$$S_1, S_2 \longrightarrow T,$$

$S_1 \times_T S_2$ is quasi-compact.

Thm 3.1 Γ induces an equivalence of categories

$$\left\{ \begin{matrix} \text{compact Hausdorff} \\ \text{spaces} \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \text{quasi-compact and quasi-separated} \\ \text{condensed sets} \end{matrix} \right\}$$

1. If X is compact Hausdorff, $\underline{X}$ is quasi-compact

Proof: $\hat{X} \twoheadrightarrow X$ (Stone-Čech)

$$\Rightarrow \quad \underline{\hat{X}} \twoheadrightarrow \underline{X} \quad \checkmark$$

(Why is this surjective?

$$\begin{tikzcd} \underline{\hat{X}} \arrow[r] & \underline{X} \\ S' \arrow[u, dashed] \arrow[r, two heads] & S \arrow[u] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & \hat{X} \arrow[r, two heads] & X \\ \widehat{\hat{X} \times_X S} \arrow[ur] \arrow[r, two heads] \arrow[d, equal] & \hat{X} \times_X S \arrow[u] \arrow[r, two heads] & S \arrow[u] \\ S' & & \end{tikzcd}$$

)

2. If X is cg and weakly Hausdorff, $\underline{X}$ is quasi-separated. In particular, if X is compact Hausdorff, $\underline{X}$ is quasi-compact quasi-separated.

Proof:

$$\begin{tikzcd} S_1 \times_{\underline{X}} S_2 \arrow[r] \arrow[d] & S_2 \arrow[d] \\ S_1 \arrow[r] & \underline{X} \end{tikzcd}$$

$$S_1 \times_{\underline{X}} S_2 \cong \underline{S_1 \times_X S_2}$$

Thus, STS that $S_1 \times_X S_2$ is compact Hausdorff.

$$\begin{tikzcd} S_1 \times_X S_2 \arrow[r, "\text{proper}"] \arrow[d, "\text{proper}"'] & S_2 \arrow[d] \\ S_1 \arrow[r] & X \end{tikzcd}$$

Since X is weakly Hausdorff, $S_1, S_2 \to X$ are proper.

$\Rightarrow$ $S_1 \times_X S_2 \to S_i$ are proper $\overset{S_1 \text{ is compact}}{\Longrightarrow}$ $S_1 \times_X S_2$ is compact (+Hausdorff because $S_1 \times S_2$ is, and it has the subspace toplogy)

~~$\Rightarrow$ $S_1 \times_X S_2 \to S_1 \times S_2$ is proper~~

~~$\Rightarrow$ $S_1 \times_X S_2$ is closed in $S_1 \times S_2$~~

~~$\Rightarrow$ $S_1 \times_X S_2$ is compact Hausdorff~~

$\Box$

3. Say $\underline{X}$ is quasi-compact and quasi-separated.

Then ∃ a surjection

$$\underline{S} \to \underline{X} \qquad S \in *_{\text{proét}}$$

evaluate on $*$ $\Rightarrow$

$$S \twoheadrightarrow X$$

$\Rightarrow$ X is compact.

To check that X is Hausdorff, STS $S \times_X S$ is closed in $S \times S$.

$S \times S$ profinite, so STS $S \times_X S$ is compact. Now,

$$\underline{S \times_X S} = \underline{S} \times_{\underline{X}} \underline{S} \quad \text{is quasi-compact}$$

$$\Rightarrow S \times_X S \text{ is compact.} \qquad \square$$

This proves the theorem. ("The pro-étale topology for schemes")

Thm 3.2: The fully faithful inclusion $\big(\overbrace{\text{Bhatt-Scholze 15}}$, 4.3.7$\big)$

$$\left\{ \begin{matrix} \text{weakly Hausdorff} \\ \text{cg spaces} \end{matrix} \right\} \hookrightarrow \left\{ \begin{matrix} \text{quasi-separated} \\ \text{condensed sets} \end{matrix} \right\}$$

induces an equivalence

$$\mathrm{Ind}\left(\{\text{ind compact Hausdorff spaces}\}\right) \xrightarrow{\sim} \left\{ \begin{matrix} \text{quasi-separated} \\ \text{condensed sets} \end{matrix} \right\}$$

$$\rotatebox{90}{$\in$}$$

$$\mathbb{CP}^\infty = \varinjlim_{N} \mathbb{CP}^N$$

Cor: If T is quasi-separated and $T(*) \simeq *$, then $T \simeq *$.

Pf/ $T = \underline{X}$ for X ind-compact. Then

$$X = \underline{X}(*) = T(*) \simeq * . \qquad \square$$

So quasi-separated things are "determined" by their "points" $T(*)$.

Ex: Consider $\mathbb{R}$.

$$0 \to \mathbb{R}^{disc} \to \mathbb{R} \to \underbrace{Q}_{\text{cokernel in Cond(Ab)}} \to 0$$

sheafification of $\left(S \mapsto \mathbb{R}(S)/\mathbb{R}^{disc}(S)\right)$

Claim: Q is not quasi-separated.

Proof: Evaluating on S gives a long exact sequence

$$0 \to \mathrm{Hom}_{Top}(S, \mathbb{R}^{disc}) \to \mathrm{Hom}_{Top}(S, \mathbb{R}) \to Q(S) \to H^1(\mathbb{R}^{disc}, S) \to \cdots$$

Thm: For T profinite and M a discrete abelian group, $H^1(T, M) = 0$.

So, $Q(S) = \mathrm{Hom}_{Top}(S, \mathbb{R}) / \mathrm{Hom}_{Top}(S, \mathbb{R}^{disc})$

Take e.g. $S = \mathbb{R}$: $Q(\mathbb{R}) \neq *$

$Q(pt) = *$.

$\Longrightarrow$ Q not quasi-separated. ☐

So we need to include these nasty non-qs condensed sets to get an abelian category. So we must leave ind-compact Hausdorff spaces.