Solid modules over $A_\square$ and $(A,\mathbb{Z})_\square$ & the exceptional pushforward

Plan

(1) Motivation

(2) Theorem Statements

(3) Proofs of Theorems $\leftarrow$ fairly involved

1 Motivation

Key Motivation Generalize coherent duality to non-proper morphisms using Solid modules

$\underbrace{\text{coherent duality}}_{\text{Poincaré duality for schemes}}$; $\underbrace{\text{proper}}_{\sim \text{compact in topology}}$

most familiar form

Serre Duality k field, X/k smooth proper d-dimensional

$$\omega_{X/k} := \Omega^d_{X/k} \quad \text{dualizing sheaf}$$

(1) There is a natural trace map $tr_{X/k} : H^d(X;\omega_{X/k}) \longrightarrow k$. $\rightsquigarrow \int_X$

(2) For all $E \in Coh(X)$, the pairing

$$H^i(X;E) \otimes_k Ext^{d-i}_X(E,\omega_{X/k}) \longrightarrow H^d(X;\omega_{X/k}) \xrightarrow{tr_{X/k}} k$$

is perfect.

Reformulation

$$H^{d-i}(X;\underline{Hom}_X(E,\omega_{X/k})) \cong Hom_k(H^i(X;E),k)$$

$$RHom_X(E,\omega_{X/k})[d] \simeq RHom_k(R\Gamma(X;E),k)$$

> Write $f : X \longrightarrow \mathrm{Spec}(k)$

$$R\mathcal{H}om_X(E, \omega_{X/k})[d] \simeq R\mathcal{H}om_k(Rf_*(E), k).$$

Recall from topology For Poincaré Duality for noncompact manifolds, we need to use compactly supported cohomology

$$Rf_* \rightsquigarrow f_! \text{ exceptional pushforward}$$

This will exist on the level of solid modules

Goal for rest of Scholze's Notes Show that in a very general setting, $f_! : D(\mathcal{O}_{X,\square}) \longrightarrow D(R_\square)$ exists, and that without properness there is a trace

$$tr_{X/R} : f_! \omega_{X/R}[d] \longrightarrow R$$

and duality equivalence

$$R\mathcal{H}om_{X_\square}(E, \omega_{X/R})[d] \xrightarrow{\sim} R\mathcal{H}om_{R_\square}(f_!(E), R).$$

Notation $(A, A[-]^\wedge, \alpha : A[-] \to A[-]^\wedge)$ instead of $(\underline{\mathcal{A}}, \mathcal{A}, \underline{\mathcal{A}} \to \mathcal{A})$.

$$\mathrm{Solid}(A^\wedge) \hookrightarrow \mathrm{Mod}^{\mathrm{cond}}(A).$$

$$D(A^\wedge) := D(\mathrm{Solid}(A^\wedge)).$$

Toward the Exceptional Pushforward $f_!$

Goal The affine, absolute case. Given a finitely gen. discrete ring A, construct

$$f_! : D(A_\square) \longrightarrow D(\mathbb{Z}_\square).$$

haven't yet shown $A_\square$ is analytic

Recall A finitely gen. ring

$$\begin{array}{ll} (A,\mathbb{Z})_\square & := \left(A,\ S \mapsto \mathbb{Z}_\square[S] \underset{\mathbb{Z}}{\otimes} A,\ A[S] \xrightarrow{\text{can}} \mathbb{Z}_\square[S] \underset{\mathbb{Z}}{\otimes} A\right) \\ \quad \Big\downarrow \text{can}_A & \\ A_\square & := \left(A,\ S \longmapsto \lim_{i\in I} A[S_i],\ A[S] \xrightarrow{\text{can}} \lim_{i\in I} A[S_i]\right) \end{array}$$

Motivation for $f_!$ To construct exceptional pushforwards for nonproper morphisms $f : X \longrightarrow Y$, try to factor

$$\begin{tikzcd} X \arrow[r, hook, "j", "\text{open}"'] \arrow[dr, "f"'] & \overline{X} \arrow[d, "\overline{f}"] \\ & Y \end{tikzcd}$$

'compactification'

proper

Non-obvious to show independence of choices!

$$f_! := \overline{f}_* j_!$$

For us $(A,\mathbb{Z})_\square$ is the compactification!

Proof later

Lemma Let A be a finitely gen. ring. Then $A_\square$ is analytic and the morphism of preanalytic rings

$$\mathrm{can}_A : (A,\mathbb{Z})_\square \longrightarrow A_\square$$

is a morphism of analytic rings.

Consequence We have a left adjoint

$$j^* := (-) \overset{L}{\underset{(A,\mathbb{Z})_\square}{\otimes}} A_\square : D((A,\mathbb{Z})_\square) \longrightarrow D(A_\square)$$

no a priori reason to be a right adjoint

Write $j_*: D(A_\square) \longrightarrow D((A,\mathbb{Z})_\square)$ for the right adjoint given by forgetting the $(A,\mathbb{Z})_\square$-module structure.

Why $(A,\mathbb{Z})_\square$ is a compactification

Key Point Solid$((A,\mathbb{Z})_\square)$ is supposed to be an enlargement of the category of quasicoherent sheaves on the adic spectrum $\mathrm{Spa}(A,\tilde{\mathbb{Z}})$.

Integral closure of $\mathrm{im}(\mathbb{Z} \to A)$

Why $\mathrm{Spa}(A,\tilde{\mathbb{Z}})$? $\mathrm{Spa}(A,\tilde{\mathbb{Z}})$ is a 'universal' (and functorial) compactification of $\mathrm{Spec}(A)$. However, $\mathrm{Spa}(A,\tilde{\mathbb{Z}})$ is an adic space, not just a scheme.

Defining $\mathrm{Spa}(A,\tilde{\mathbb{Z}})$ We want a factorization

$$\begin{tikzcd} \mathrm{Spec}(A) \arrow[r, hook, "j", "\text{open}"'] \arrow[dr, "f"'] & \mathrm{Spa}(A,\tilde{\mathbb{Z}}) \arrow[d, "\bar{f}", "\text{proper}"'] \\ & \mathrm{Spec}(\mathbb{Z}) \end{tikzcd}$$

also called $\mathrm{Spv}(A)$, the Valuation Spectrum

in a universal way.

> Use the valuative criterion for properness to define the points of the 'compactification' $\mathrm{Spa}(A,\tilde{\mathbb{Z}})$.

Not the most general version, but least complicated

Valuative Criterion for Properness $p: X \to Y$ morphism of finite type between locally Noetherian schemes. Then p is proper if and only if for every valuation ring V and commutative square

$$\begin{tikzcd} \operatorname{Spec}(\operatorname{Frac}(V)) \arrow[r] \arrow[d, hook, "\text{open}"'] & X \arrow[d, "p"] \\ \operatorname{Spec}(V) \arrow[r] \arrow[ur] & Y \end{tikzcd},$$

there exists a unique lift $\operatorname{Spec}(V) \longrightarrow X$ making the diagram commute.

Point We should define points of our desired compactification $\operatorname{Spa}(A, \tilde{\mathbb{Z}})$ to be commutative squares

$$\begin{tikzcd} \operatorname{Spec}(\operatorname{Frac}(V)) \arrow[r] \arrow[d, hook] & \operatorname{Spec}(A) \arrow[d] \\ \operatorname{Spec}(V) \arrow[r] & \operatorname{Spec}(\mathbb{Z}) \end{tikzcd}$$

these $\longrightarrow$ are not actually extra data, but become relevant in the relative setup.

> This set is quotiented out by the equivalence relation **generated** by the following relation: given a surjection of spectra of valuation rings

$$\mathrm{Spec}(W) \twoheadrightarrow \mathrm{Spec}(V)$$

(i.e., faithfully flat map), we say that the elements defined by the right-hand square and outer rectangle in the diagram

$$\begin{tikzcd}
\mathrm{Spec}(\mathrm{Frac}(W)) \arrow[r] \arrow[d, hook] & \mathrm{Spec}(\mathrm{Frac}(V)) \arrow[r] \arrow[d, hook] & \mathrm{Spec}(A) \arrow[d] \\
\mathrm{Spec}(W) \arrow[r, two heads] & \mathrm{Spec}(V) \arrow[r] & \mathrm{Spec}(\mathbb{Z})
\end{tikzcd}$$

are equivalent.

$$\mathrm{Spa}(A, \tilde{\mathbb{Z}}) := \left\{ \begin{tikzcd}
\mathrm{Spec}(\mathrm{Frac}(V)) \arrow[r] \arrow[d, hook] & \mathrm{Spec}(A) \arrow[d] \\
\mathrm{Spec}(V) \arrow[r] & \mathrm{Spec}(\mathbb{Z})
\end{tikzcd} \right\} \Big/ \sim$$

- Then one has to put a topology on $\mathrm{Spa}(A, \tilde{\mathbb{Z}})$ as well as a sheaf of rings.
- One can then show that $f: \mathrm{Spec}(A) \to \mathrm{Spec}(\mathbb{Z})$ factors as a composite of maps of locally ringed spaces

Similarly There is an adic Spectrum $\mathrm{Spa}(A, A)$ defined by replacing A by $\mathbb{Z}$:

$$\mathrm{Spa}(A,A) := \{\mathrm{Spec}(V) \longrightarrow \mathrm{Spec}(A)\}/\sim .$$

> There is a map

$$\mathrm{Spa}(A,A) \xrightarrow{\mathrm{can}_A} \mathrm{Spa}(A,\tilde{\mathbb{Z}})$$

$$\left[\mathrm{Spec}(V) \xrightarrow{\phi} \mathrm{Spec}(A)\right] \longmapsto \left[\begin{tikzcd} \mathrm{Spec}(\mathrm{Frac}(V)) \arrow[r, "\phi j"] \arrow[d, hook, "j"'] & \mathrm{Spec}(A) \arrow[d, "f"] \\ \mathrm{Spec}(V) \arrow[r, "f\phi"'] & \mathrm{Spec}(\mathbb{Z}) \end{tikzcd}\right].$$

Observation Write

$$\mathrm{Spa}(A,A)_{\mathrm{triv}} \subset \mathrm{Spa}(A,A) \quad \text{and} \quad \mathrm{Spa}(A,\tilde{\mathbb{Z}})_{\mathrm{triv}} \subset \mathrm{Spa}(A,\tilde{\mathbb{Z}})$$

for the equivalence classes with a representative where V is a field (i.e., a rank 0 valuation ring). Note that:

(1) The map can_A restricts to a bijection

$$\mathrm{can}_A : \mathrm{Spa}(A,A)_{\mathrm{triv}} \xrightarrow{\sim} \mathrm{Spa}(A,\tilde{\mathbb{Z}})_{\mathrm{triv}}$$

(2) The map

$$\begin{aligned} \mathrm{Spec}(A) &\longrightarrow \mathrm{Spa}(A,A) \\ \mathfrak{p} &\longmapsto [\mathrm{Spec}(\kappa(\mathfrak{p})) \longrightarrow \mathrm{Spec}(A)] \end{aligned}$$

is injective with image $\mathrm{Spa}(A,A)_{triv}$.

(3) There are retractions

$$\begin{aligned} \mathrm{Spa}(A,A) &\xrightarrow{\quad r \quad} \mathrm{Spa}(A,A)_{triv} \cong \mathrm{Spec}(A) \\ [\mathrm{Spec}(V)\to\mathrm{Spec}(A)] &\longmapsto [\mathrm{Spec}(\mathrm{Frac}(V)) \longrightarrow \mathrm{Spec}(A)] \end{aligned}$$

and

$$\mathrm{Spa}(A,\tilde{\mathbb{Z}}) \longrightarrow \mathrm{Spa}(A,\tilde{\mathbb{Z}})_{triv} \cong \mathrm{Spec}(A)$$

$$\left[\begin{tikzcd} \mathrm{Spec}(\mathrm{Frac}(V)) \arrow[r] \arrow[d, hook] & \mathrm{Spec}(A) \arrow[d] \\ \mathrm{Spec}(V) \arrow[r] & \mathrm{Spec}(\mathbb{Z}) \end{tikzcd}\right] \longmapsto \left[\begin{tikzcd} \mathrm{Spec}(\mathrm{Frac}(V)) \arrow[r] \arrow[d, equal] & \mathrm{Spec}(A) \arrow[d] \\ \mathrm{Spec}(\mathrm{Frac}(V)) \arrow[r] & \mathrm{Spec}(\mathbb{Z}) \end{tikzcd}\right].$$

Theorem There is a fully faithful functor

$$\mathrm{Sch}^{\mathrm{noeth}} \longrightarrow \{\text{Adic Spaces}\}$$

that sends $\mathrm{Spec}(A)$ to $\mathrm{Spa}(A,A)$. Moreover, there is an isomorphism of locally ringed spaces

$$(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}) \cong (\mathrm{Spa}(A,A)_{\mathrm{triv}}, r_*\mathcal{O}_{\mathrm{Spa}(A,A)}).$$

Upshot We get a factorization

$$\begin{tikzcd}
\mathrm{Spec}(A) \arrow[r, hook] & \mathrm{Spa}(A,A) \arrow[rr, hook, "\mathrm{can}_A", "\text{open}"'] \arrow[rrd, "f"'] & & \mathrm{Spa}(A,\tilde{\mathbb{Z}}) \arrow[d, "\bar{f}", "\text{proper}"'] \\
& & & \mathrm{Spa}(\mathbb{Z},\mathbb{Z}) \arrow[d, "r"] \\
& & & \mathrm{Spec}(\mathbb{Z})
\end{tikzcd}$$

2 Theorem Statements

Theorem 1 A finitely generated $\mathbb{Z}$-algebra.

(1.1) j^* admits a $\underbrace{\text{fully faithful}}_{\Leftrightarrow\, j_* \text{ ff}}$ left adjoint $\underbrace{j_! : D(A_\square) \hookrightarrow D((A,\mathbb{Z})_\square)}$.

$$\begin{tikzcd} D(A_\square) \arrow[r, hook, shift left=2, "j_!"] \arrow[r, hook, shift right=2, "j_*"'] & D((A,\mathbb{Z})_\square) \arrow[l, "j^*"' description] \end{tikzcd}$$

(1.2) **Projection formula** for $M \in D((A,\mathbb{Z})_\square)$

$$j_! j^*(M) \simeq M \otimes_{(A,\mathbb{Z})_\square} j_!(A).$$

Def Let $f : \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(\mathbb{Z})$ be a finitely gen. $\mathbb{Z}$-algebra. Write

$$f_! : D(A_\square) \xhookrightarrow{j_!} D((A,\mathbb{Z})_\square) \xrightarrow{\text{forget}} D(\mathbb{Z}_\square).$$

> Since $j_!$ and the forgetful functor are left adjoints, $f_!$ admits a right adjoint $f^! : D(\mathbb{Z}_\square) \longrightarrow D(A_\square)$.

Theorem 2 $f: \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(\mathbb{Z})$ finitely gen. $\mathbb{Z}$-algebra. Then:

(2.1) $f^!$ is a left adjoint.

$\Updownarrow$ Since $D(A_\square)$ and $D(\mathbb{Z}_\square)$ are compactly generated

(2.2) $f_!$ preserves compact objects

(2.3) Projection formula for all $M \in D(\mathbb{Z}_\square)$, $N \in D(A_\square)$

$$f_!\left(\left(M \overset{L}{\underset{\mathbb{Z}_\square}{\otimes}} A_\square\right) \overset{L}{\underset{A_\square}{\otimes}} N\right) \simeq M \overset{L}{\underset{\mathbb{Z}_\square}{\otimes}} f_!(N).$$

Theorem 3 $f: \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(\mathbb{Z})$ finitely gen. $\mathbb{Z}$-algebra. Then:

(3.1) $f^!: D(\mathbb{Z}_\square) \longrightarrow D(A_\square)$ is given by $f^!(M) \simeq \left(M \overset{L}{\underset{\mathbb{Z}_\square}{\otimes}} A_\square\right) \overset{L}{\underset{A_\square}{\otimes}} f^!(\mathbb{Z})$.

(3.2) $f^!(\mathbb{Z})$ is a bounded complex of discrete A-modules.

(3.3) $f^!: D(\mathbb{Z}_\square) \longrightarrow D(A_\square)$ preserves discrete objects. $\Longleftarrow$ (3.1) + (3.2)

(3.4) If f is a complete intersection, then $f^!(\mathbb{Z}) \in D(A)$ is invertible.

3 Proofs of Theorems

Initial Reduction Can reduce to $A = \mathbb{Z}[t]$.

(1) Since A is f.g., we can choose a surjection

$$\mathbb{Z}[t_1, \dots, t_n] \twoheadrightarrow A.$$

A simple base change argument lets us reduce to $A = \mathbb{Z}[t_1, \dots, t_n]$.

(2) An inductive argument lets us reduce to $n = 1$.

Key Idea $f_!(A)$ can be computed as 'functions near the boundary' of $\mathrm{Spec}(A)$.

$$A := \mathbb{Z}[t] \quad , \quad A_\infty := \mathbb{Z}(\!(t^{-1})\!)$$

> We'll show $j_!(A) \simeq (\mathbb{Z}(\!(t^{-1})\!) / \mathbb{Z}[t])[-1]$

$$f^!(\mathbb{Z}) \simeq \mathbb{Z}[t][1]$$

Goal Once we know that $A_\square$ is analytic, we know

$$D(A_\square) \subset D(\mathrm{Mod}^{\mathrm{cond}}(A)) \supset D((A,\mathbb{Z})_\square)$$

We then want to:

(1) Show that the forgetful functor $j_*: D(A_\square) \to D((A,\mathbb{Z})_\square)$ is an inclusion.

(2) Provide an embedding $D(A_\infty) \hookrightarrow D((A,\mathbb{Z})_\square)$

(3) Show that $D((A,\mathbb{Z})_\square)$ is the recollement of $D(A_\square)$ with $D(A_\infty)$.

$$\begin{tikzcd}
D(A_\square) \arrow[r, bend left, "j_!"] \arrow[r, bend right, "j_*"'] & D((A,\mathbb{Z})_\square) \arrow[l, "j^*"'] \arrow[r, bend left, "(-)\otimes^{L}_{(A,\mathbb{Z})_\square} A_\infty"] \arrow[r, bend right, "R\mathrm{Hom}(A_\infty,-)"'] & D(A_\infty) \arrow[l, hook']
\end{tikzcd}$$

In fact, (2) and (3) will be used to show $A_\square$ is analytic.

Step 2

Recall Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a symmetric monoidal ∞-category. A commutative algebra R is $\mathcal{C}$ is **idempotent** if the multiplication $R \otimes R \to R$ is an equivalence.

> In this case, being an R-module is a **property**: the forgetful functor $\mathrm{Mod}_R(\mathcal{C}) \to \mathcal{C}$ is fully faithful with image those $X \in \mathcal{C}$ such that

$$Y \otimes \mathbb{1} \xrightarrow{\mathrm{id} \otimes \mathrm{unit}} Y \otimes R$$

is an equivalence.

Point Want to show that A_∞ is idempotent in $D((A, \mathbb{Z})_\square)$.

Observation 1 There is a short exact sequence

$$0 \longrightarrow \mathbb{Z}[[u]] \otimes_{\mathbb{Z}} \mathbb{Z}[t] \xrightarrow{ut-1} \mathbb{Z}[[u]] \otimes_{\mathbb{Z}} \mathbb{Z}[t] \longrightarrow \underset{\underset{A_\infty}{\parallel}}{\mathbb{Z}((t^{-1}))} \longrightarrow 0$$

Consequence 2 $\mathbb{Z}[[u]] \otimes_{\mathbb{Z}} \mathbb{Z}[t]$ compact projective $\Longrightarrow A_\infty$ is compact in $\mathrm{Solid}((A, \mathbb{Z})_\square)$.

Consequence 3 Using this presentation of A_∞, we see that

$$\text{mult}: A_\infty \overset{L}{\underset{(A,\mathbb{Z})_\square}{\otimes}} A_\infty \xrightarrow{\sim} A_\infty$$

Consequence 4

$$\begin{tikzcd} \underbrace{\mathrm{Mod}_{A_\infty}(D((A,\mathbb{Z})_\square))}_{M \xrightarrow{\sim} M \overset{L}{\underset{(A,\mathbb{Z})_\square}{\otimes}} A_\infty} \arrow[r, hook] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & D((A,\mathbb{Z})_\square) \arrow[l, bend right, "(-) \overset{L}{\otimes} A_\infty"'] \arrow[l, bend left, "R\mathrm{Hom}(A_\infty,-)"] \end{tikzcd}$$

Lemma 5 Let $C_* \in D(\mathrm{Mod}^{\mathrm{cond}}(A))$ be such that each C_i is a direct sum of products of copies of A. Then

$$R\mathrm{Hom}_A(A_\infty, C_*) \simeq 0.$$

Proof

Since $D((A,\mathbb{Z})_\square) \subset D(\mathrm{Mod}^{\mathrm{cond}}(A))$ is closed under limits & colimits:

(1) By writing $C_* \simeq \lim_n \underbrace{C_{*\geq n}}_{\text{brutal truncation}}$, can assume C_* is connective.

(2) Since A_∞ is compact, by writing C_* as a filtered colimit, suffices to treat the case $C_* = \prod_I A$.

$\rightsquigarrow$ reduces to $C_* = A$.

By Observation 1,

$$\mathrm{RHom}_A(A_\infty, A) \simeq \left[\mathrm{RHom}_{\mathbb{Z}}(\mathbb{Z}[\![u]\!], A) \xrightarrow{ut-1} \mathrm{RHom}_{\mathbb{Z}}(\mathbb{Z}[\![u]\!], A) \right]$$

$$\simeq \left[A[u^{-1}]/A \xrightarrow{ut-1} A[u^{-1}]/A \right]$$

$$\simeq \left[\mathbb{Z}[u^{-1}] \xrightarrow[\sim]{-1} \mathbb{Z}[u^{-1}] \right]. \leftarrow \text{acyclic}$$

$\square$

Lemma 6 For any set I,

$$\mathrm{Coker}\left(A \otimes_{\mathbb{Z}} \prod_I \mathbb{Z} \hookrightarrow \prod_I A \right) \in D(A_\infty)$$

> Need this fact + Lemma 6 to see that $A_\square$ is analytic.

Proof

$$\begin{tikzcd}
\mathbb{Z}[t] \otimes_{\mathbb{Z}} \prod_I \mathbb{Z} \arrow[r, hook] \arrow[d, hook] & \prod_I \mathbb{Z}[t] \arrow[r] \arrow[d, hook] & \mathrm{Coker}_1 \arrow[r] \arrow[d, "\wr"] & 0 \\
\mathbb{Z}((t^{-1})) \otimes_{\mathbb{Z}[[t^{-1}]]} \prod_I \mathbb{Z}[[t^{-1}]] \arrow[r, hook] \arrow[d, hook] & \prod_I \mathbb{Z}((t^{-1})) \arrow[r] \arrow[d] & \mathrm{Coker}_2 \arrow[r] & 0 \\
\prod_I t^{-1}\mathbb{Z}[[t^{-1}]] \arrow[r, "\sim"] \arrow[d] & \prod_I t^{-1}\mathbb{Z}[[t^{-1}]] \arrow[d] & & \\
0 & 0 & &
\end{tikzcd}$$

Sequence of $\mathbb{Z}((t^{-1}))$-modules

$\square$

Proof that $\mathbb{Z}[t]_\square$ is analytic

Let $C_* \in D(\mathrm{Mod}^{\mathrm{cond}}(A))$ be such that each C_i is a direct sum of products of copies of A.

Need to show For S extremally disconnected,

$$R\mathcal{H}om_A(A[S], C_*) \simeq R\mathcal{H}om_A(A_\square[S], C_*).$$

Since $C_* \in D((A,\mathbb{Z})_\square)$, we know that

$$R\mathcal{H}om_A(A[S], C_*) \simeq R\mathcal{H}om_A((A,\mathbb{Z})_\square[S], C_*).$$

Since $\mathbb{Z}_\square[S] \cong \prod_I \mathbb{Z}$ for some set I, we have

$$(A,\mathbb{Z})_\square[S] \cong A \otimes_{\mathbb{Z}} \prod_I \mathbb{Z} \quad \text{and} \quad A_\square[S] \cong \prod_I A.$$

So we need to see that

$$R\mathcal{H}om_A\left(A \otimes_{\mathbb{Z}} \prod_I \mathbb{Z}, C_*\right) \simeq R\mathcal{H}om_A\left(\prod_I A, C_*\right).$$

Equivalently, that

$$\operatorname{RHom}_A\Big(\underbrace{\operatorname{Coker}\Big(A \underset{\mathbb{Z}}{\otimes} \prod_I \mathbb{Z} \hookrightarrow \prod_I A\Big)}_{A_\infty\text{-module by Lemma 6}}, C_*\Big) \simeq 0$$

By Lemma 5

$\operatorname{RHom}_A(A_\infty, C_*) \simeq 0$.

$\square$

Proof of Theorem 1

Lemma 7

$$\operatorname{Ker}\left(j^* : D((A,\mathbb{Z})_\square) \longrightarrow D(A_\square) \right) = D(A_\infty)$$

Proof

- $M \in D(A_\infty) \Longrightarrow j^*(M)$ is a module over $\underbrace{A_\infty \overset{L}{\underset{(A,\mathbb{Z})_\square}{\otimes}} A_\square}_{= j^*(A_\infty)} \simeq 0$. ← Lemma 5

- $\operatorname{Ker}(j^*)$ is generated by the A_∞-modules ← Lemma 6

$$\operatorname{Coker}\left(A \underset{\mathbb{Z}}{\otimes} \prod_I \mathbb{Z} \hookrightarrow \prod_I A \right)$$

$\square$

Recollection on Recollements

Definition Let X be a stable ∞-category and

$$i_*: Z \hookrightarrow X \quad \text{and} \quad j_*: U \hookrightarrow X$$

stable subcategories. We say X is the recollement of (Z,U) if:

(1) $i_*: Z \hookrightarrow X$ admits a left adjoint $i^*: X \to Z$.

(2) $j_*: U \hookrightarrow X$ admits a left adjoint $j^*: X \to U$.

(3) The composite $Z \xhookrightarrow{i_*} X \xrightarrow{j^*} U$ is zero.

(4) The functors $i^*: X \to Z$ and $j^*: X \to U$ are jointly conservative.

Lemma In this situation:

(1) i_* admits a right adjoint $i^!: X \to Z$ defined by

$$i_* i^! := \mathrm{fib}\left(\mathrm{id}_X \xrightarrow{\text{unit}} j_* j^* \right)$$

(2) j^* admits a $\underbrace{\text{fully faithful}}^{\text{free since } j_* \text{ is ff.}}$ left adjoint $j_!: U \hookrightarrow X$ defined by

$$j_! j^* := \mathrm{fib}\left(\mathrm{id}_X \xrightarrow{\text{unit}} i_* i^*\right)$$

(3) $j_* : U \hookrightarrow X$ identifies U with the right orthogonal complement

$$Z^{\perp} := \{ x \in X \mid \forall z \in Z,\ \mathrm{Map}_X(i_*(z), x) \simeq 0 \}$$

(4) $j_! : U \hookrightarrow X$ identifies U with the left orthogonal complement

$$^{\perp}Z := \{ x \in X \mid \forall z \in Z,\ \mathrm{Map}_X(x, i_*(z)) \simeq 0 \}.$$

Picture

$$\begin{tikzcd}
Z \arrow[r, hook, "i_*" description] & X \arrow[l, bend right=40, "i^*"'] \arrow[l, bend left=40, "i^!"] \arrow[r, "j^*" description] & U \arrow[l, hook, bend right=40, "j_!"'] \arrow[l, hook, bend left=40, "j_*"]
\end{tikzcd}$$

Lemma Assume we are given adjunctions of stable ∞-categories:

$$\begin{tikzcd} Z \arrow[r, hook, shift right, "i_*"'] & X \arrow[l, shift right, "i^*"'] \arrow[r, shift left, "j^*"] & U \arrow[l, hook, shift left, "j_*"] \end{tikzcd}$$

The following are equivalent:

(1) X is the recollement of (Z, U).

(2) $Z \simeq \mathrm{Ker}(j^* : X \longrightarrow U)$.

Back to Theorem 1

Our Situation

$$\begin{tikzcd} D(A_\infty) \arrow[rr, hook] & & D((A,\mathbb{Z})_\square) \arrow[ll, bend right, "(-)\overset{L}{\otimes} A_\infty"'] \arrow[ll, bend left, "R\mathrm{Hom}(A_\infty,-)"] \arrow[rr, "{j^* = (-)\overset{L}{\otimes} A_\square}"] & & D(A_\square) \arrow[ll, hook, "j_*"] \end{tikzcd}$$

and $D(A_\infty) = \mathrm{Ker}(j^*)$. By the Lemma:

- There is an extreme fully faithful left adjoint $j_! : D(A_\square) \hookrightarrow D((A,\mathbb{Z})_\square)$. Computing using the formula for $j_!$, we see that

$$j_! j^*(M) := \mathrm{fib}\left(M \overset{L}{\underset{(A,\mathbb{Z})_\square}{\otimes}} A_\square \longrightarrow M \overset{L}{\underset{(A,\mathbb{Z})_\square}{\otimes}} A_\infty \right)$$

$$\simeq M \overset{L}{\underset{(A,\mathbb{Z})_\square}{\otimes}} \mathrm{fib}(A \longrightarrow A_\infty)$$

$$\simeq M \overset{L}{\underset{(A,\mathbb{Z})_\square}{\otimes}} A/A_\infty[-1].$$

Projection Formula (1.2)

$$j_! j^*(M) \simeq M \otimes_{(A,\mathbb{Z})_\square} j_!(A)$$

Immediate from the definition of $j_!$. $\square$

Proof of Theorem 2

Proof that $f_!$ preserves compact objects (2.2)

The objects $\prod_I A$ form a collection of compact generators for $D(A_\square)$. Computing:

$$
\begin{aligned}
j_!\left(\prod_I A\right) &\simeq j_! j^*\left(A \underset{\mathbb{Z}}{\otimes} \prod_I \mathbb{Z}\right) \\
&\simeq \left(A \underset{\mathbb{Z}}{\otimes} \prod_I \mathbb{Z}\right) \underset{(A,\mathbb{Z})_\square}{\otimes} (A_\infty / A)[-1] \\
&\simeq \prod_I \mathbb{Z} \underset{\mathbb{Z}_\square}{\otimes} (A_\infty / A)[-1] \\
&\simeq \prod_I (A_\infty / A)[-1]
\end{aligned}
$$

Prop. 6.3

$$\left(\prod_I \mathbb{Z}\right) \overset{L}{\underset{\mathbb{Z}_\square}{\otimes}} \left(\prod_J \mathbb{Z}\right) \simeq \prod_{I \times J} \mathbb{Z}$$

Now note that

$$A_\infty / A = \mathbb{Z}((t^{-1})) / \mathbb{Z}[t]$$

is compact in $D(\mathbb{Z}_\square)$. $\square$

Proof of Projection Formula (2.3)

We want to show that for all $M \in D(\mathbb{Z}_\square)$ and $N \in D(A_\square)$,

$$f_!\left(\left(M \overset{L}{\underset{\mathbb{Z}_\square}{\otimes}} A_\square\right) \overset{L}{\underset{A_\square}{\otimes}} N\right) \simeq M \overset{L}{\underset{\mathbb{Z}_\square}{\otimes}} f_!(N).$$

For this, it suffices to prove the more refined formula

(*) $$j_!\left(\left(M \overset{L}{\underset{\mathbb{Z}_\square}{\otimes}} A_\square\right) \overset{L}{\underset{A_\square}{\otimes}} N\right) \simeq \left(M \overset{L}{\underset{\mathbb{Z}_\square}{\otimes}} A_\square\right) \overset{L}{\underset{\mathbb{Z}_\square}{\otimes}} j_!(N).$$

Note that:

(1) After $(-) \otimes A_\infty$, both terms vanish.

(2) After applying j^*, both sides are the same.

The recollement of $D((A,\mathbb{Z})_\square)$ into $D(A_\square)$ and $D(A_\infty)$ shows that (*) holds. $\square$

Proof of Theorem 3

Proof that $f^!(\mathbb{Z})$ is discrete + invertible (3.2) + (3.4)

$$
\begin{aligned}
f^!(\mathbb{Z}) &\simeq R\mathrm{Hom}_{\mathbb{Z}}(f_!(\mathbb{Z}[t]), \mathbb{Z}) \quad \text{adjunction} \\
&\simeq R\mathrm{Hom}_{\mathbb{Z}}((\mathbb{Z}((t^{-1}))/\mathbb{Z}[t])[-1], \mathbb{Z}) \\
&\simeq R\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}((t^{-1}))/\mathbb{Z}[t], \mathbb{Z})[1] \\
&\simeq \mathbb{Z}[t][1] \quad \text{cofiber sequence for } \mathbb{Z}((t^{-1}))/\mathbb{Z}[t]
\end{aligned}
$$

$\square$

Proof of formula for $f^!$ (3.1)

We want to show

$$
f^!(M) \simeq \left(M \overset{L}{\underset{\mathbb{Z}_\square}{\otimes}} A_\square\right) \overset{L}{\underset{A_\square}{\otimes}} \underbrace{f^!(\mathbb{Z})}_{A[1]}
$$

Note that by adjunction

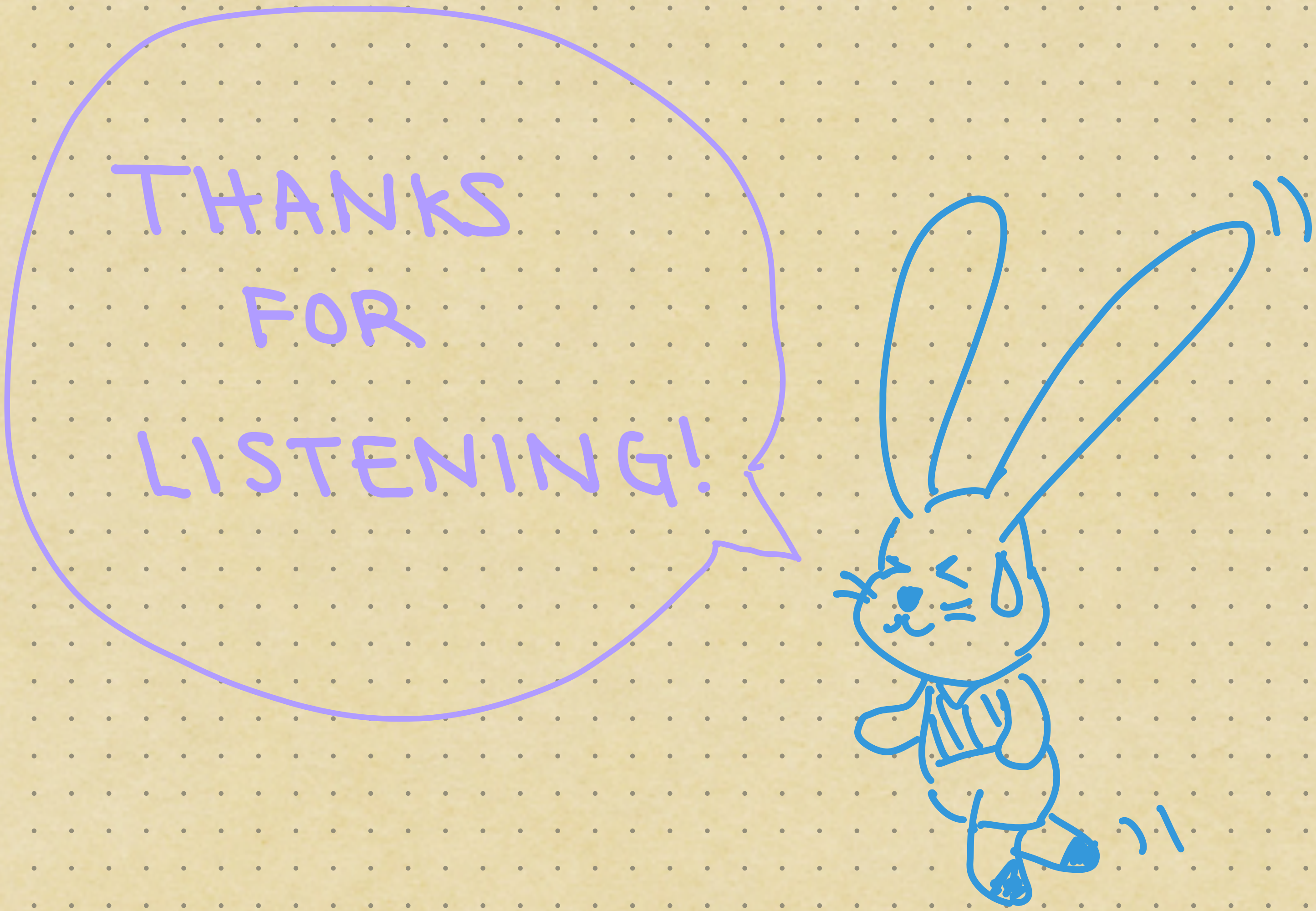
$$
f^!(M) \longleftarrow \left(M \overset{L}{\underset{\mathbb{Z}_\square}{\otimes}} A_\square\right) \overset{L}{\underset{A_\square}{\otimes}} f^!(\mathbb{Z})
$$

$$\begin{tikzcd}
M & f_!\left(\left(M \overset{L}{\underset{\mathbb{Z}_\square}{\otimes}} A_\square\right) \overset{L}{\underset{A_\square}{\otimes}} f^!(\mathbb{Z})\right) \arrow[l] \arrow[dl, "\sim" description, "\text{projection formula}"] \\
M \underset{\mathbb{Z}_\square}{\otimes} f_! f^!(\mathbb{Z}) \arrow[u, "M \otimes \text{counit}"] &
\end{tikzcd}$$

To see this is an equivalence, note:

(1) Both sides preserve colimits, so we can reduce to the case where $M = \prod_I \mathbb{Z}$ is a compact generator.

(2) Since $f^!$ commutes with products and both agree when $M = \mathbb{Z}$, the claim follows by noting that the RHS commutes with products. $\square$


THANKS
FOR
LISTENING!