3/9 Def. of Condensed Sets.

1\. To get a notion of space carrying "topology" on which we can conveniently do algebra. take hint from

Fact : Sheaves of abelian groups on a topological space form an abelian category. More generally, same statement holds if replacing "topological space" by ~~any~~ "site".

[03CM]

Giving a full proof would be too much a digression, let's content in reviewing the concepts and applying intuition for top. sp..

Def (Site) A site C is a category with a family of fixed targets $\{U_i \to U\}_{i \in I}$ called coverings, denoted as $Cov(C)$, satisfying.

(1) isomorphisms are coverings

(2) coverings of coverings are coverings:

$(\{V_i \to U\}_i \in Cov(C), \{V_{ij} \to V_i\}_j \in Cov(C))$

$\Rightarrow \{V_{ij} \to U\}_{i,j} \in Cov(C)$

(3) product of a covering is a covering:

$(\{V_i \to U\}_i \in Cov(C), W \to U$ a morph.$)$

$\Rightarrow (\{V_i \times_U W \to W\}_i$ exist and $\in Cov(C))$

(Note, we don't require all products exist, only those of coverings)

Classical ~~Alan~~ examples: $Op(X)$, $Ét(X)$.

Def (presheaf) Let C be a site.

· A presheaf (of sets) on C is a functor $C^{op} \to Set$.

· A sheaf (of sets) on C is a presheaf $\mathcal{F}: C^{op} \to Set$ satisfying: $\forall \{U_i \to U\} \in Cov(C)$,

$$\mathcal{F}(U) \longrightarrow eq\left(\prod_i \mathcal{F}(U_i) \overset{diff}{\underset{0}{\rightrightarrows}} \prod_{i,j} \mathcal{F}(U_i \times_U U_j)\right) \text{ is an iso.}$$

Remarks

① Avoid thinking about "points" of a site — they are mostly useless (for us).
Question: how to do sheafification without points?
[00W1] **Answer:** FACT: $\mathcal{F} \in \mathrm{Presh}(\mathcal{C})$. $\mathcal{F}^{+} := U \mapsto \operatorname{colim}_{\{U_i \to U\}_i \in \mathrm{Cov}(\mathcal{C})} \operatorname{eq}\left(\prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_i \times_U U_j)\right)$
Then $\mathcal{F}^{++} \in \mathrm{Sh}(\mathcal{C})$.

② Most intuitions from top sp. carry over, e.g.:
[03CM] a) $\mathrm{Ab}(\mathcal{C})$ is abelian. Actually much more (to be discussed in later talks) (e.g. [01DL] enough inj., lim., colim, exist)
b) $\otimes$ in $\mathrm{Ab}(\mathcal{C})$: $\mathcal{F} \otimes \mathcal{G} :=$ sheafification of $U \mapsto \mathcal{F}(U) \otimes \mathcal{G}(U)$.
Hom: $\mathcal{H}om(\mathcal{F}, \mathcal{G}) :=$ sheafification of $U \mapsto \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$.
(restriction means viewing $\mathcal{F}$ as in $\mathrm{Ab}(\mathcal{C}/U)$.)
c) a useful construction:
$\mathcal{F} \in$ ~~Ab(C)~~ $\mathrm{Sh}(\mathcal{C})$, $\mathbb{Z}[\mathcal{F}] \in \mathrm{Ab}(\mathcal{C})$ defined as
$(U \longmapsto \mathbb{Z}[\mathcal{F}(U)])^{++}$ (free ab. gp. over $\mathcal{F}(U)$).
This is left adjoint to forget: $\mathrm{Ab}(\mathcal{C}) \longrightarrow \mathrm{Sh}(\mathcal{C})$.
It is symmetric monoidal ($\times$ ..... $\otimes$).

2. So if we define "topological" abelian groups as sheaves of abelian groups on some site then we automatically get a nice category.

Def. (condensed objects) Let ProFin be the site of profinite sets with coverings given by finite jointly surjective maps.
Cond(Set) := Sheaves of Sets on ProFin.
ab. gp./rings/modules ... ab. gp./rings/modules ...

Question: Why ProFin? Why this covering?
Answer: a) The whole theory serves as a justification.
b) We will say more about the covering later.

[BS 15] c) Historical note: ProFin arose as the pro-étale site of a (geometric) point in Bhatt-Scholze's theory.

Remarks: ① We think of condensed sets/ab. gp./… as sets (spaces)/ab. gp./… carrying "topology".
Define spaces as sheaves is nothing new: Schemes are special sheaves on Ring,
[Question: make the word "special" precise.]
and more generally, algebraic spaces.

[Condensed, App II] ② There are set-theoretic issues. We ignore.

~~Let's study the definition.~~

[BH 19] ③ Barwick-Haine's work is closely related.

Let's study the definition.

①: ~~Examples of~~ profinite sets:

~~Def~~ A profinite set is a limit of finite discrete sets. $S = \varprojlim_{i\in I} S_i$

There is a natural topology on S: subspace topology of product ~~topology~~. By Tychonoff, S is compact, Hausdorff.

More explicitly, consider the case $I = \overset{1}{\bullet}\leftarrow\overset{2}{\bullet}\leftarrow\overset{3}{\bullet}\cdots$

Then a basis of topology is given as follows:

$S(A_n) ~~A_1,\dots,A_n~~ := \{(a_1, a_2, \dots) \in \prod S_i \mid a_n \in A_n\}$

where $n \in \mathbb{Z}_{\geq 1}$, A_n is an arbitrary subsets of S_n.

It is clear from this description that S is totally disconnected. General I is similar.

[Recall: tot. dis. $\Leftrightarrow$ conn. components are pts. $\Leftrightarrow$ any subset containing more than ~~two~~ one pt. is disconn.]

(Caution, comp. always closed but not necessarily open.)

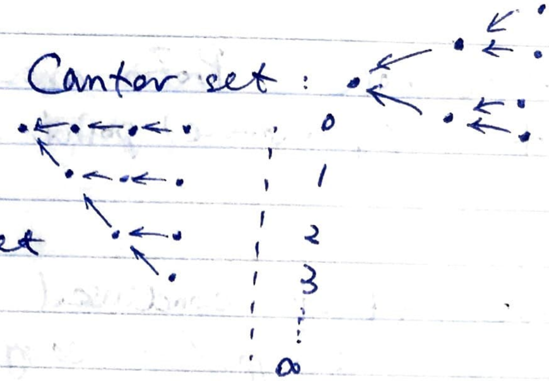
[08ZY]. FACT: · A top. sp. is a profinite sp. $\Longleftrightarrow$ it is compact, Hausdorff, tot. dis.

· I can always be taken to be cofiltered.

Examples: · Cantor set:

· $\mathbb{N} \cup \{\infty\}$:

· any finite set



($\cong \mathbb{Z}_2$, FACT: $\mathbb{Z}_p \cong$ Cantor)

[Question: $\mathbb{Q}_p$?] [countable union of Cantor]

[Question: non Cantor profinite fractals?]

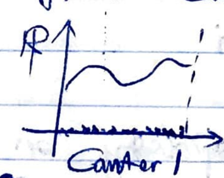
Maps

· maps in Top $\mathbb{N} \cup \{\infty\}$ to a top. sp. X are convergent sequences in X.

· ~~maps in Top from a profin. S to a discrete set X are $\operatorname{colim} \operatorname{Map}(S_i, X)$ because 1) S is compact so image is compact hence finite 2) $\forall z \in$ Image, its inverse image is open hence contains some basic open of the form $S(A_{n_z})$, n_z depending on z. 3) as z ranges over the image, let $N \in I$ larger than all n_z, then~~

~~proof: 1) S compact so image is compact hence finite 2) ! $\square$~~

· maps in Top from Cantor to ($\mathbb{R}$, usual) can be quite arbitrary:



Locally const.

· maps in Top from a profin. S to any target X $= \operatorname{colim} \operatorname{Map}(S_i, X)$.

proof: Each pt in S has an open nbhd on which the map is const. This gives a covering of S, by compactness finitely many of them covers. We may assume they are of the form $S(A_\alpha)$ $\alpha \in \{i_1, \dots, i_n\}$. Let $m \in I$ s.t. $m > i_1, \dots, i_n$. Then the map determines, and is determined by a map $\in \operatorname{Map}(S_m, X)$. $\square$

② **The sheaf condition**: can be reduced to

i) $\mathcal{F}(\phi) = *$

ii) $\mathcal{F}(S_1 \amalg S_2) \longrightarrow \mathcal{F}(S_1) \times \mathcal{F}(S_2)$ is iso.

iii) $\forall\ S' \twoheadrightarrow S$,

$$\mathcal{F}(S) \longrightarrow \{x \in \mathcal{F}(S') \mid p_1^*(x) = p_2^*(x) \in \mathcal{F}(S' \times_S S')\} \text{ is iso.}$$

proof: In other words, need to show sheaf condition can be reduced to i) $U = \phi$ ii) $\{S_1, S_2\} \twoheadrightarrow S_1 \amalg S_2$ iii) $S' \twoheadrightarrow S$.

[The simplicity of the sheaf condition partially justifies our choice of coverings.

Ex(ercise)/(ample) $\forall X \in Top$. $\mathcal{F} \in Presh(ProFin) : S \mapsto Map(S, X)$ is a sheaf.

Suffices to observe, $\forall \{U_i \to U\}_{i \in I}$ finite,

$$\prod_i \mathcal{F}(U_i) = \mathcal{F}(\textstyle\coprod_i U_i) \text{ by ii)},$$

So $$\prod_{i,j} \mathcal{F}(U_i \times_U U_j) = \mathcal{F}(\textstyle\coprod_{i,j} U_i \times_U U_j) = \mathcal{F}\big((\coprod_i U_i) \times_U (\coprod_j U_j)\big)$$

So general seq. follows from seq. for $(\coprod_i U_i) \to U$. $\square$

[MilneEtale Coh. II 1.5] (This is similar and simpler than a similar statement for étale sheaves.)

3. It is convenient to formulate two equivalent def. of condensed objects.

General ~~topology~~ preliminaries:

i) extremally disconnected sets :⇔ closure of any open is open.

[Gleason 58] FACT A compact Hausdorff space S is ~~extdis~~ ext. dis. $\Leftrightarrow$ any $S' \twoheadrightarrow S$, $S' \in CHaus$, splits.

Denote them as ExtDis. Then we have

$$CHaus \supseteq ProFin \supseteq ExtDis. \quad \text{as categories.}$$

FACT ExtDis does not have product.

ii) Stone–Čech compactification of a top. sp. X := a compact Hausdorff sp. βX together with $X \xrightarrow{i} \beta X$, s.t. $\forall K$ compact, $f : X \to K$, $\exists!$ factorisation

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & \beta X \arrow[d, dashed] \\ & K \end{tikzcd}$$

It always exists.

• For X discrete, $\beta X \in ExtDis$, because

$$\begin{tikzcd} & S \arrow[d, two heads] \\ X \arrow[ur, dashed] \arrow[r, "i"] & \beta X \arrow[u, dashed, bend right] \end{tikzcd}$$

* This implies any $S \in CHaus$ receives a surjection from an object in ExtDis, namely βS^δ where S^δ is S with discrete top.

$$\left(\begin{tikzcd} S^\delta \arrow[r, two heads] \arrow[d] & S \\ \beta S^\delta \arrow[ur] & \end{tikzcd}\right)$$

Now consider CHaus ⊇ ProFin ⊇ ExtDis.
using finite jointly surjective maps as coverings, CHaus, ProFin become sites.
note: ExtDis is not a site.

~~Prop: Any functor from ExtDis^{op} to a fixed category $\mathcal{C}$ extends uniquely to a functor from ProFin to $\mathcal{C}$~~

Prop Let $\mathcal{C}$ be the category of sets/gps./rings.

① Any functor $\mathcal{F} : \text{ExtDis}^{op} \to \mathcal{C}$ s.t.

- $\mathcal{F}(\emptyset) = *$
- $\mathcal{F}(S_1 \amalg S_2) \to \mathcal{F}(S_1) \times \mathcal{F}(S_2)$ is iso.

extends uniquely and functorially to a sheaf on ProFin taking values in $\mathcal{C}$.

② Any sheaf of ProFin valued in $\mathcal{C}$ extends uniquely and functorially to a sheaf of CHaus valued in $\mathcal{C}$.

proof (sketch):

②: for $S \in$ CHaus, $\mathcal{F}(S)$ is determined as follows: by FACT ii), $\exists\, S' \twoheadrightarrow S$, $S' \in$ ProFin.
Then $\mathcal{F}(S) = \text{eq}\left(\mathcal{F}(S') \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \mathcal{F}(S' \times_S S')\right)$.

①: for $S \in$ ProFin, $\mathcal{F}(S)$ is determined as follows: by FACT ii), $\exists\, S' \twoheadrightarrow S$, $S' \in$ ExtDis, $\exists\, S'' \twoheadrightarrow S' \times_S S'$, $S'' \in$ ExtDis.
Then $\mathcal{F}(S) = \text{eq}\left(\mathcal{F}(S') \rightrightarrows \mathcal{F}(S'')\right)$
where the two maps are pullbacks via

$$\begin{tikzcd} S'' \arrow[dr, two heads] & & \\ & S' \times_S S' \arrow[r] \arrow[d] & S' \arrow[d] \\ & S' \arrow[r, two heads] & S \end{tikzcd}$$

(note different sections on $S' \times_S S'$ always pull back to different sections on S'') □